Question 1 (12 marks)

(a) Define boundary of a set.

(b) Define interior and closure of a set.

(c) Let $X = \mathbb{R}$ with the usual topology. Determine (with proof) $\partial([0,1])$.

(d) Let $X = \mathbb{R}$ with the usual topology. Determine $\partial\mathbb{Q}$ (with proof, of course).

Question 2 (15 marks)

(a) Define uniformly continuous.

(b) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$ be a function. Show that $f: X \to Y$ is uniformly continuous if and only if f satisfies

$$\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that}$$
$$\text{if } x, y \in X \text{ and } d(x,y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon.$$

Question 3 (10 marks)

(a) Define inner product space.

(b) Define the length norm.

(c) Let $(V, \langle , \rangle)$ be an inner product space. Show that if $x, y \in V$ then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Question 4 (20 marks)

(a) Define metric space.

(b) Define metric space topology.

(c) Define continuous.

(d) Let X and Y be metric spaces and let $f: X \to Y$ be a function. Let $a \in X$. Show that f is continuous at a if and only if f satisfies:

$$\text{if } \varepsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that } f(B_\delta(a)) \subseteq B_\varepsilon(f(a)).$$

Question 5 (25 marks)

(a) Define contraction.

(b) Carefully state the fixed point theorem for contractions.

(c) Which of the following maps are contractions (with proof)?

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x}$;
(2) $f : [0, \infty) \to [0, \infty)$, $f(x) = e^{-x}$;
(3) $f : [0, \infty) \to [0, \infty)$, $f(x) = e^{-e^{x}}$;
(4) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$;
(5) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(\cos x)$.

Question 6 (16 marks)

(a) Let $a, b \in \mathbb{R}$ with $a < b$. Show that if $f\colon [a, b] \to \mathbb{R}$ is a function such that $f\colon [a, b] \to \mathbb{R}$ is continuous and $f'\colon (a, b) \to \mathbb{R}$ exists then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(c)(b - a).$$

(b) Carefully state the intermediate value theorem.

Question 7 (20 marks) Let C be the circle in $\mathbb{R}^2$ with the centre at $(0, 1/2)$ and radius $1/2$. Let $X = C \setminus \{(0, 1)\}$. Define the function $f : \mathbb{R} \to X$ by defining $f(t)$ to be the point at which the line segment from $(t, 0)$ to $(0, 1)$ intersects X.

(a) Show that $f : \mathbb{R} \to X$ and $f^{-1} : X \to \mathbb{R}$ are continuous.

(b) Define topologically equivalent metric spaces.

(c) Define $\rho\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$\rho(s, t) = |f(s) - f(t)|$$

where $|\ |$ is the standard norm in $\mathbb{R}^2$. Show that ρ defines a metric on $\mathbb{R}$.

(d) Show that ρ is topologically equivalent to the standard metric on $\mathbb{R}$.

Question 8 (18 marks)

(a) Define metric space.

(b) Define ℓ^∞.

(c) Define separable.

(d) Show that ℓ^∞ is a metric space and ℓ^∞ is not separable.

End of Exam—Total Available Marks = 136

Question 1 (10 marks)

(a) Define metric space.

(b) Define $B_\epsilon(x)$.

(c) Define closure.

(d) Give an example of a metric space (X, d) and a point $x \in X$ such that

$$\overline{B_1(x)} \neq \{y \in X \mid d(y, x) \leq 1\}.$$

Question 2 (15 marks) Let $a \in \mathbb{R}_{>0}$ and let

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \quad \text{for } x \in \mathbb{R}_{>0}.$$

(a) Show that $f(x) \geq \sqrt{a}$ for $x \in \mathbb{R}_{>0}$. Hence f defines a function $f\colon X \to X$ where $X = [\sqrt{a}, \infty)$.

(b) Show that f is a contraction mapping when X is given the usual metric.

(c) Fix $x_0 > \sqrt{a}$ and $x_{n+1} = f(x_n)$ for all $n \geq 0$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

Question 3 (20 marks) Let $(X_1, d_1), \ldots, (X_\ell, d_\ell)$ be metric spaces. Show that a sequence $\vec{x}_n = (x_n^{(1)}, \ldots, x_n^{(\ell)})$ in $X_1 \times \cdots \times X_\ell$ converges if and only if each of the sequences $x_n^{(i)}$ (in X_i) converges.

Question 4 (20 marks)
Let $X = [0, 2\pi)$ and $Y = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $f\colon [0, 2\pi) \to S^1$ be given by

$$f(x) = (\cos x, \sin x).$$

(a) Show that f is continuous.

(b) Show that f is a bijection.

(c) Show that $f^{-1}\colon S^1 \to [0, 2\pi)$ is not continuous.

(d) Why does this not contradict the following statement: *Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Assume f is a bijection, X is compact and Y is Hausdorff. Then the inverse function $f^{-1}\colon Y \to X$ is continuous.*

Question 5 (12 marks)

(a) Define topological space.

(b) Define connected.

(c) Let X be a topological space. Let $\mathcal{S}$ be a collection of connected subsets of X such that $\bigcap_{A\in\mathcal{S}} A \neq \emptyset$. Show that $\bigcup_{A\in\mathcal{S}} A$ is connected.

Question 6 (15 marks)
Let (X, d) be a metric space and let $a \in X$.

(a) Define uniformly continuous for metric spaces.

(b) Show that

$$\text{if } x, y \in X \text{ then } \quad |d(x,a) - d(y,a)| \leq d(x,y).$$

(c) Show that the function $f\colon X \to \mathbb{R}$ defined by $f(x) = d(x,a)$ is uniformly continuous.

Question 7 (10 marks)

(a) Define normed vector space.

(b) Define the standard norm on $\mathbb{R}^n$ and show that $\mathbb{R}^n$, with this norm, is a normed vector space.

Question 8 (16 marks)

(a) Define bounded linear operator.

(b) Define compact linear operator.

(c) Define self adjoint linear operator.

(d) Define $V^\perp$.

(e) Let T be a bounded self adjoint compact operator on a Hilbert space H. Assume λ is a non zero complex number so that $\lambda I - T$ is an surjective function. Use the fact that

$$\text{if } N = \ker(\lambda I - T) \text{ and } R = \overline{\operatorname{im}(\lambda I - T)} \text{ then } N = R^\perp$$

to prove that $\lambda I - T$ is one-to-one and has a bounded inverse.

Question 9 (12 marks)

(a) Define positive operator.

(b) Prove that if T is a positive operator then every eigenvalue of T is non-negative.

End of Exam—Total Available Marks = 130

Question 1 (12 marks)

(a) Define topological space.

(b) Define continuous function.

(c) Define compact subset.

(d) Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous function. Let $K \subseteq X$. Show that if K is compact then $f(X)$ is compact.

Question 2 (14 marks)

(a) Define connected.

(b) Define interval.

(c) Define closed subset.

(d) Define bounded.

(e) Provide a detailed sketch of the proof of the following statement. Let $A \subseteq \mathbb{R}$, where the metric on $\mathbb{R}$ is given by $d(x, y) = |x - y|$. Show that

$$A \text{ is connected and compact} \quad \text{if and only if} \quad A \text{ is a closed and bounded interval.}$$

Question 3 (20 marks)

(a) Carefully define continuous and uniformly continuous functions for metric spaces.

(b) Let $n \in \mathbb{Z}_{>0}$. Prove that the function $x^n\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.

(c) Let $n \in \mathbb{Z}_{>1}$. Prove that the function $x^n\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is not uniformly continuous.

(d) Let $n \in \{0, 1\}$. Prove that the function $x^n\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is uniformly continuous.

Question 4 (20 marks)

Let $X = \mathbb{Q}$ with the usual metric and let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q}$. For $n \in \mathbb{Z}_{>0}$ let $Q_n = \mathbb{Q} - \{q_n\}$.

(a) Show that if $n \in \mathbb{Z}_{>0}$ then Q_n is open and dense.

(b) Show that $\displaystyle\bigcap_{n \in \mathbb{Z}_{>0}} Q_n = \emptyset$.

(c) Carefully state the Baire theorem for open dense sets.

(d) Explain (with proof) why parts (a) and (b) do not provide a counterexample to the Baire theorem on open dense sets.

Question 5 (18 marks)
Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint linear operator.

(a) Show that there exists $x \in H$ with $\|x\| = 1$ and $|\langle Tx, x\rangle| = \|T\|$.

(b) Let $x \in H$ be as in (a). Show that x is an eigenvector of T with eigenvalue $\|T\|$.

(c) Use the proof of (a) to explicitly produce an eigenvector of the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & \pi \\ -2 & \pi & 0 \end{pmatrix}$$

Question 6 (10 marks)
Let H be a Hilbert space and let $T\colon H \to H$ be a nonzero bounded compact self adjoint linear operator.

(a) Define compact linear opearator.

(b) Provide a detailed sketch of the proof that there exists an orthonormal basis of H consisting of eigenvectors of T.

Question 7 (12 marks)

(a) Define normed vector space.

(b) Define metric space.

(c) Define the norm metric.

(d) Let V be a normed vector space. Prove that V with the norm metric is a metric space.

Question 8 (10 marks) Let V, W be closed subspaces of a Hilbert space H.

(a) Define Hilbert space.

(b) Define closed subspace.

(c) Prove that if $W \perp V$ then $W + V$ is closed.

Question 9 (10 marks) Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ with the subspace topology. Let $f\colon X \to Y$ be a continuous function.

(a) Define the subspace topology.

(b) Show that

$$g\colon \begin{array}{ccc} A & \to & Y \\ a & \mapsto & f(a) \end{array} \quad \text{is continuous.}$$

End of Exam—Total Available Marks = 126

Question 1.

Let (X,d) be a metric space and let $A \subseteq X$.

10pts (a) Define the different kinds of compactness (including "closed" and "bounded").

20pts (b) Draw the diagram relating the different kinds of compactness.

20pts (c) Choose one of the implications in your diagram and prove it.

Question 2.

10pts (a) Carefully define a topology.

10pts (b) Carefully define a metric space.

10pts (c) Carefully define the ball of radius ϵ centred at x.

20pts (d) Let (X,d) be a metric space. Define

$$\mathcal{B} = \{B_\epsilon(x) \mid x \in X, \epsilon \in \mathbb{R}_{\geq 0}\}$$

and let

$$\mathcal{T} = \left\{ U \subseteq X \;\middle|\; \text{there exists } \mathcal{R} \subseteq \mathcal{B} \text{ such that } U = \bigcup_{B \in \mathcal{R}} B \right\}$$

Show that $\mathcal{T}$ is a topology on X.

Question 3.

20pts (a) Let $n \in \mathbb{Z}_{>0}$. Prove that the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ is continuous.

20pts (b) Prove that the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$ is continuous.

20pts (c) Let $f \colon S \to T$ be a function. Prove that the inverse function to f exists if and only if f is bijective.

Question 4.

10pts (a) Carefully define a uniformity.

10pts (b) Carefully define a metric space.

10pts (c) Carefully define the diagonal of width radius ϵ.

20pts (d) Let (X,d) be a metric space and let

$$\mathcal{X} = \{U \subseteq X \times X \mid \text{there exists } \epsilon \in \mathbb{R}_{>0}\; U \supseteq B_\epsilon\}$$

Show that $\mathcal{T}$ is a uniformity on X.

Question 5.

Assume that it is known that $\mathbb{R}_{\geq 0}$ is complete.

10pts (a) Prove that if $A \subseteq \mathbb{R}_{\geq 0}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists.

10pts (b) Give an example (with proof) of an increasing sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geq 0}$ which does not converge.

10pts (c) Give an example (with proof) of a bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geq 0}$ which does not converge.

10pts (d) Prove that if $(a_1, a_2, \ldots)$ is an increasing and bounded sequence in $\mathbb{R}_{\geq 0}$ then $(a_1, a_2, a_3, \ldots)$ converges.

10pts (e) Give an example (with proof) of an increasing and bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{Q}_{\geq 0}$ which does not converge.

Question 6.

Let (X, d) be a metric space and let $(a_1, a_2, \ldots)$ be a sequence in X.

10pts (a) Carefully define cluster point and limit point of $(a_1, a_2, \ldots)$.

10pts (b) Prove that if z is a limit point of $(a_1, a_2, \ldots)$ then z is a cluster point of $(a_1, a_2, \ldots)$.

10pts (c) Carefully define Cauchy sequence and convergent sequence.

10pts (d) Prove that if $(a_1, a_2, \ldots)$ converges then $(a_1, a_2, \ldots)$ is Cauchy.

10pts (e) Carefully define complete metric space.

Question 7.

10pts (a) Carefully define a "topology on X" and a "uniformity on X".

10pts (b) Let (X, d) be a metric space. Carefully define the "metric space topology on X" and the "metric space uniformity on X".

10pts (c) Determine all the topologies on the set $X = \{0, 1\}$.

10pts (d) Determine all the uniformities on $X = \{0, 1\}$.

10pts (e) For each of the uniformities you gave in part (d), compute the uniform space topology.

Question 8.

10pts (a) Carefully define a normed vector space.

10pts (b) Carefully define a positive definite Hermitian inner product space.

10pts (c) Carefully state and prove the Cauchy-Schwarz inequality.

10pts (d) Carefully state and prove the Pythagorean theorem.

10pts (e) Carefully state and prove the parallelogram law.