Assignment 1

MAST30026 Metric and Hilbert Spaces Semester II 2017
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to be turned in before 2pm on 7 September 2017

(1) (diameters and distance to A) Let (X, d) be a metric space. Let $A \subseteq X$ with $A \neq \emptyset$. The *diameter of* A is

$$\mathrm{diam}(A) = \sup\{d(x,y) \mid x, y \in A\} \qquad \text{and} \qquad d(x,A) = \inf\{d(x,a) \mid a \in A\}$$

is the *distance from* x *to* A.

(a) Let A and B be bounded subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that
$$\mathrm{diam}(A \cup B) \leqslant \mathrm{diam}(A) + \mathrm{diam}(B).$$
What can you say if A and B are disjoint?

(b) Prove that $\overline{A} = \{x \in X \mid d(x, A) = 0\}$.

(c) Prove that if $x, y \in X$ then $|d(x, A) - d(y, A)| \leqslant d(x, y)$.
[Hint: first show that $d(x, A) \leqslant d(x, y) + d(y, A)$.]

(d) Show that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.

(e) Assume $x \in X$ and $x \notin \overline{A}$. Let
$$U = \{y \in X : d(y, A) < d(x, A)\}.$$
Show that U is an open set in X, $U \supseteq \overline{A}$ and $x \notin U$.

(2) (product topology) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{T}$ be the product topology on $X \times Y$. Let $\mathcal{N}(p)$ denote the neighborhood filter of p.

(a) Provide an example (with proof) of topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ and an open set Z in $X \times Y$ such that there do not exist open sets U in X and V in Y with $Z = U \times V$.

(b) Let $x \in X$ and $y \in Y$. Show that
$$\mathcal{N}((x,y)) = \{Z \subseteq X \times Y \mid \text{there exist } U \in \mathcal{N}(x) \text{ and } V \in \mathcal{N}(y) \text{ with } Z \supseteq U \times V\}.$$

(c) Show that if $A \subseteq X$ and $B \subseteq Y$, then $\overline{A} \times \overline{B} = \overline{A \times B}$.

(d) Prove that if X and Y are path connected, then $X \times Y$ is also path connected.

(3) (Uniform spaces are almost metric spaces) Let X be a set. A *pseudometric on* X is a function $d\colon X \times X \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that

(a) If $x \in X$ then $d(x,x) = 0$,

(b) If $x, y \in X$ then $d(x,y) = d(y,x)$,

(c) If $x, y, z \in X$ then $d(x,y) \leqslant d(x,z) + d(z,y)$.

For $\epsilon \in \mathbb{R}_{>0}$ let $B_\epsilon = \{(x_1, x_2) \in X \times X \mid d(x_1, x_2) \leqslant \epsilon\}$.

For $E \subseteq X \times X$ let $\sigma(E) = \{(y,x) \mid (x,y) \in E\}$.

(a) Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a pseudometric on X. Let

$$\mathcal{X} = \{V \subseteq X \times X \mid \text{there exists } \epsilon \in \mathbb{R}_{>0} \text{ such that } V \supseteq B_\epsilon\}.$$

Show that $\mathcal{X}$ is a uniformity on X.

(b) Let $(X, \mathcal{X})$ be a uniform space. Let $E \in \mathcal{X}$ and $x \in X$. Carefully define the E-neighborhood of x and the neighborhood filter of x and prove that

$$\mathcal{N}(x) = \{B_E(x) \mid E \in \mathcal{X}\}.$$

(c) Let $(X, \mathcal{X})$ be a uniform space. For $E \in \mathcal{X}$, choose $E_1, E_2, \ldots \in \mathcal{X}$ such that

$$\sigma(E_n) = E_n, \quad E_1 \subseteq E, \quad \text{and} \quad E_{n+1} \times_X E_{n+1} \subseteq E_n.$$

and let

$$\mathcal{X}_E = \{D \subseteq X \times X \mid \text{there exists } k \in \mathbb{Z}_{>0} \text{ with } D \supseteq E_k\}.$$

(ca) Show that $\mathcal{X}_E$ is a uniformity on X and $\mathcal{X}_E \subseteq \mathcal{X}$.

(cb) Show that $\mathcal{X} = \sup\{\mathcal{X}_E \mid E \in \mathcal{X}\}$.

(d) Let $E \in \mathcal{X}$, let $E_1, E_2, \ldots$ be as in (c) and let $U_1, U_2, \ldots \in \mathcal{X}_E$ such that

$$\sigma(U_n) = U_n, \quad U_1 \subseteq E_1, \quad \text{and} \quad U_{n+1} \times_X U_{n+1} \times_X U_{n+1} \subseteq U_n \cap E_n.$$

Define $g_E\colon X \times X \to \mathbb{R}$ by

$$g_E(x,y) = \begin{cases} 1, & \text{if } (x,y) \notin U_1, \\ 2^{-k}, & \text{if } (x,y) \in U_1, (x,y) \in U_2, \ldots, (x,y) \in U_k \text{ and } (x,y) \notin U_{k+1}, \\ 0, & \text{if } (x,y) \in U_n \text{ for } n \in \mathbb{Z}_{>0}. \end{cases}$$

Show that

$$g_E(x,y) = g_E(y,x), \quad g_E(x,y) \in \mathbb{R}_{\geqslant 0}, \quad \text{and} \quad \text{if } x \in X \text{ then } g_E(x,x) = 0.$$

(e) Define $d_E\colon X \times X \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ by

$$d_E(x,y) = \inf\{g_E(x,z_1) + \cdots + g_E(z_{p-1},y) \mid p \in \mathbb{Z}_{>0},\ z_1, \ldots, z_{p-1}, z_p \in X,\ z_p = y\}.$$

(ea) Show that if $x, y \in X$ then $\frac{1}{2}g_E(x,y) \leqslant d_E(x,y) \leqslant g_E(x,y)$.

(eb) Show that d_E is a pseudometric.

(ec) Show that the uniformity defined by d_E (as in part (b)) is equal to $\mathcal{X}_E$.

(f) Give an example (with proof) of a set X and a pseudometric on X which is not a metric on X.

(4) (Continuous and uniformly continuous functions)

(a) Show that the composition of continuous functions is continuous.

(b) Show that the composition of uniformly continuous functions is uniformly continuous.

(c) Give an example of a bijective continuous function such that the inverse function is not continuous.

(5) (posets and topological spaces) Let X be a set. A *preorder on* X is a relation $\leqslant$ on X such that

(A) If $a \in X$ then $a \leqslant a$,

(B) If $a, b, c \in X$ and $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$.

Let $(X, \leqslant_X)$ and $Y, \leqslant_Y)$ be preordered sets. A *monotone function* is a function $f\colon X \to Y$ such that

$$\text{if } x_1, x_2 \in X \text{ and } x_1 \leqslant x_2 \text{ then } f(x_1) \leqslant f(x_2).$$

(a) Let $X = \{1, 2, 3\}$. Carefully describe all preorders on X and all topologies on X.

(b) Let $(X, \leqslant)$ be a preordered set.

$$\mathcal{T}_X = \{U \subseteq X \mid \text{if } x \in U \text{ and } y \in X \text{ and } x \leqslant y \text{ then } y \in U\}.$$

Show that $\mathcal{T}$ is a topology on X.

(c) Let $(Y, \mathcal{T})$ be a topological space. Define a relation $\leqslant$ on Y by

$$x \leqslant y \qquad \text{if } x \in \overline{\{y\}}$$

where $\overline{A}$ is the closure of A. Show that $\leqslant$ is a preorder on Y.

(d) Define a function $\mathcal{F}\colon \{\text{topological spaces}\} \to \{\text{preordered sets}\}$ by

$$\mathcal{F}((Y, \mathcal{T})) = (Y, \leqslant), \qquad \text{where } \leqslant \text{ is as defined in part (b).}$$

Show that if $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a continuous function then $f\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ is monotone.

(e) Define a function $\mathcal{G}\colon \{\text{preordered sets}\} \to \text{topological spaces}\}$ by

$$\mathcal{G}((X, \leqslant)) = (X, \mathcal{T}), \qquad \text{where } \mathcal{T} \text{ is as defined in part (c).}$$

Show that if $f\colon (X, \leqslant_X) \to (Y, \leqslant_Y)$ is monotone then $f\colon \mathcal{G}(X) \to \mathcal{G}(Y)$ is continuous.

(f) Show that if $(X, \leqslant)$ is a preordered set then $\mathcal{F}(\mathcal{G}(X, \leqslant)) = (X, \leqslant)$.

(g) Show that if $(Y, \mathcal{T})$ is a topological space then $\mathcal{G}(\mathcal{F}(Y))$ is not necessarily equal to $(Y, \mathcal{T})$.

(h) Show that if $(Y, \mathcal{T})$ is a topological space and X is finite then $\mathcal{G}(\mathcal{F}(Y)) = (Y, \mathcal{T})$.

(6) (pointwise convergence does not imply uniform convergence) Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\}, \qquad (f_1, f_2, \dots) \text{ a sequence in } F$$

and let $f\colon X \to C$ be a function.

(a) Show that if $(f_1, f_2, \dots)$ converges uniformly to f then $(f_1, f_2, \dots)$ converges pointwise to f.

(b) Let $X = C = \mathbb{R}_{[0,1]} = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$ with metric given by $d(x, y) = \rho(x, y) = |x - y|$. For $n \in \mathbb{Z}_{>0}$ let

$$\begin{array}{rccc} f_n\colon & \mathbb{R}_{[0,1]} & \to & \mathbb{R}_{[0,1]} \\ & x & \mapsto & x^n \end{array} \qquad \text{and let } f\colon \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$$

be given by

$$f(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < 1, \\ 1, & \text{if } x = 1. \end{cases}$$

Carefully graph f_1, f_2, f_3, f_4 and f. Show that $(f_1, f_2, \dots)$ converges pointwise to f but does not converge uniformly to f.

(7) (Connected sets) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The set E is *connected* if there do not exist open sets U and V in X such that

$$U \cap A \neq \emptyset, \quad V \cap A \neq \emptyset, \qquad U \cup V \supseteq E \quad \text{and} \quad (U \cap V) \cap E = \emptyset.$$

The set E is *path connected* if E satisfies

$$\begin{array}{c} \text{if } x, y \in E \quad \text{then there exists a continuous function} \\ f\colon \mathbb{R}_{[0,1]} \to E \text{ with } f(0) = x \text{ and } f(1) = y. \end{array}$$

(a) Show that if E is path connected then E is connected.

(b) Give an example (with proof) of a connected set E which is not path connected.

(c) Let $\{0, 1\}$ have the discrete topology and let A have the subspace topology. Show that A is connected if and only if there does not exist a continuous surjective function $f\colon A \to \{0, 1\}$.

(d) Show that if $A \subseteq X$ is connected then $\overline{A}$ is connected.

(8) (Banach fixed point theorem and Picard iteration) Let (X, d) be a metric space. A *contraction mapping* is a function $f\colon X \to X$ such that there exists $\alpha \in \mathbb{R}_{>0}$ such that $\alpha < 1$ and

$$\text{if } x, y \in X \quad \text{then} \quad d(f(x), f(y)) \leqslant \alpha\, d(x, y).$$

A *fixed point* of $f\colon X \to X$ is an element $x \in X$ such that $f(x) = x$.

Picard iteration is a method for solving equations of the the form $f(x) = x$ where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. The process is to let

$$a_1 = \text{ your choice,} \quad a_2 = f(a_1), \quad a_3 = f(a_2), \quad \ldots,$$

and compute $\quad a = \lim_{n\to\infty} a_n.$

(a) Let (X, d) be a complete metric space and let $f\colon X \to X$ be a contraction mapping. Let $x \in X$ and let $x_1, x_2, \ldots$ be the sequence

$$x_1 = f(x), \quad x_2 = f(f(x)), \quad x_3 = f(f(f(x))), \quad \ldots.$$

Show that the sequence $x_1, x_2, \ldots$ converges and $p = \lim_{n\to\infty} x_n$ is the unique fixed point of f.

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $a_1 \in \mathbb{R}$ and let $a_{n+1} = f(a_n)$. Show that if the sequence $(a_1, a_2, \ldots)$ converges and $a = \lim\limits_{n\to\infty} a_n$ then $f(a) = a$.

(c) Rewrite the equation $x^3 - x - 1 = 0$ as $x = f(x)$, where $f(x) = \frac{1}{x^2+1}$. Let $a_1 = \frac{1}{2}$ and use Picard iteration to compute a solution to (5 decimal places) to $x^3 - x - 1 = 0$. Verify that your solution is correct

(d) Rewrite the equation $x^3 - x - 1 = 0$ in the form $x = f(x)$, where $f(x) = 1 - x^3$. Let $a_1 = \frac{1}{2}$ and use Picard iteration to compute a solution to (5 decimal places) to $x^3 - x - 1 = 0$. Verify that your solution is correct.

(e) Explain carefully how parts (c) and (d) provide examples and insight into the Banach fixed point theorem.

(9) (The Cantor set)

(a) Show that the Cantor set is the set of real numbers with $\frac{1}{3}$-adic expansion with no 1s.

(b) Show that $\mathrm{Card}(C) = \mathrm{Card}(\mathbb{R})$.

(c) Show that if $x \in C$ then there exists $\epsilon \in \mathbb{R}_{>0}$ such that $(x - \epsilon, x + \epsilon) \cap C = \{x\}$.

(d) Show that C is totally disconnected.

(e) Show that C is closed.

(f) Show that C is compact.