Tutorial Sheet Week of 29.08.2016

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Let $f: X \to Y$ be a continuous function.

Let $E \subseteq X$. Show that

if E is connected then $f(E)$ is connected.

Proof: To show: If $f(E)$ is not connected then E is not connected.

Assume $f(E)$ is not connected.

Then there exist $A, B \in \mathcal{T}_Y$ with

$$A \cap f(E) \neq \phi, \quad B \cap f(E) \neq \phi, \quad f(E) \subseteq A \cup B$$

$$(A \cap f(E)) \cap (B \cap f(E)) = \phi.$$

To show: E is not connected.

To show: There exist $C, D \in \mathcal{T}_Y$ with

$$C \cap E \neq \phi, \quad D \cap E \neq \phi, \quad E \subseteq C \cup D$$

$$(C \cap E) \cap (D \cap E) = \phi.$$

Let $C = f^{-1}(A)$ and $D \in f^{-1}(B)$.

To show: (a) $C \in \mathcal{T}_X$ and $D \in \mathcal{T}_X$

(b) $C \cap E \neq \phi$ and $D \cap E \neq \phi$.

(c) $E \subseteq C \cup D$

(d) $(C \cap E) \cap (D \cap E) = \phi$.

(a) Since $A, B \in \mathcal{T}_Y$ and $f: X \to Y$ is continuous then $C = f^{-1}(A) \in \mathcal{T}_X$ and $D = f^{-1}(B) \in \mathcal{T}_X$.

(b) To show: $C \cap E \neq \phi$ and $D \cap E \neq \phi$.

To show: There exists $e_1 \in C \cap E$ and $e_2 \in D \cap E$.

Since $A \cap f(E) \neq \phi$ there exists $y_1 \in A \cap f(E)$

Since $B \cap f(E) \neq \phi$ there exists $y_2 \in B \cap f(E)$.

Since $y_1, y_2 \in f(E)$ there exist $e_1, e_2 \in E$ with $f(e_1) = y_1$ and $f(e_2) = y_2$.

Since $y_1 \in A$ then $e_1 \in f^{-1}(A) = C$. So $e_1 \in C \cap E$.

Since $y_2 \in B$ then $e_2 \in f^{-1}(B) = D$. So $e_2 \in D \cap E$.

So $C \cap E \neq \phi$ and $D \cap E \neq \phi$.

(c) To show: $E \subseteq C \cup D$

To show: If $e \in E$ then $e \in C \cup D$.

Assume $e \in E$.

Then $f(e) \in f(E) \subseteq A \cup B$.

So $e \in f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = C \cup D$.

So $E \subseteq C \cup D$

(d) To show: $(C \cap E) \cap (D \cap E) = \emptyset$.

$$
\begin{aligned}
(C \cap E) \cap (D \cap E) &= C \cap D \cap E \\
&= f^{-1}(A) \cap f^{-1}(B) \cap E \\
&\subseteq f^{-1}(A) \cap f^{-1}(B) \cap f^{-1}(f(E)) \\
&= f^{-1}(A \cap f(E) \cap B \cap f(E)) \\
&= \emptyset.
\end{aligned}
$$

So E is not connected.

Thus, if E is connected then $f(E)$ is connected.