23.08.2016 (1)
M+H ~~Lect 13~~
Tutorial Sheet

<u>Proposition</u> Let $X = \mathbb{R}$ with the standard metric.
Let $A \subseteq X$.

(a) If A is bounded then A is ball compact.

(b) If A is closed in X then A is Cauchy compact.

<u>Proof of (a)</u> Assume $a \in X$ and $M \in \mathbb{R}_{>0}$ with $A \subseteq B_M(a)$.

So $A \subseteq (a-M, a+M)$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in X$ such that $A \subseteq B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_\ell)$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in X$ with $A \subseteq B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_\ell)$

Let $\ell = \frac{2M}{\varepsilon}$ and let

$$a_1 = a - M,\ a_2 = a_1 + \varepsilon,\ a_3 = a_1 + 2\varepsilon,\ a_4 = a_1 + 3\varepsilon, \ldots, a_\ell = a_1 + \ell\varepsilon.$$

Then
$$A \subseteq (a-M, a+M) \subseteq (a_1-\varepsilon, a_1+\varepsilon) \cup (a_2-\varepsilon, a_2+\varepsilon) \cup \cdots (a_\ell-\varepsilon, a_\ell+\varepsilon)$$
$$= B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_\ell)$$

So A is ball compact. //

<u>Proposition</u> Let (X,d) be a metric space such that X is Cauchy compact. Let $A \subseteq X$.

If A is closed in X then A is Cauchy compact.

23.08.2016
M+H ~~Lect 10~~ Tutorial Sheet ②

<u>Proof</u> Assume A is closed in X.

To show: A is Cauchy compact.

To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence in A
then $(a_1, a_2, \ldots)$ converges in A.

Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in A.

Then $(a_1, a_2, \ldots)$ is a Cauchy sequence in X.

Since X is complete $(a_1, a_2, \ldots)$ converges in X.

Since A is closed $\lim_{k\to\infty} a_k \in A$.

So $(a_1, a_2, \ldots)$ converges in A //.

Thus, using the assignment question which shows that $\mathbb{R}$ is complete,
then
if $A \subseteq \mathbb{R}$ and A is closed in $\mathbb{R}$
then A is Cauchy compact.