Lecture 9: Metric and Hilbert spaces 11.08.2016 (1)
Univ. Melbourne

Subspaces: $\mathbb{Z}_{\geq 0} \subseteq \mathbb{Q}_{\geq 0} \subseteq \mathbb{R}_{\geq 0}$

Can we get the topology on $\mathbb{Z}_{\geq 0}$ from the topology on $\mathbb{R}_{\geq 0}$?

Let $(X, \mathcal{T})$ be a topological space. Let $Y \subseteq X$. The subspace topology on Y is given by

$$\left\{ \begin{matrix} \text{open sets} \\ \text{in } Y \end{matrix} \right\} = \left\{ U \cap Y \mid \begin{matrix} U \text{ is open} \\ \text{in } X \end{matrix} \right\}.$$

Example $\left(\frac{1}{2}, \frac{5}{2}\right)_{\mathbb{R}_{\geq 0}} \cap \mathbb{Z}_{\geq 0} = [1,2]_{\mathbb{Z}_{\geq 0}}$, and so $[1,2]$ is open in the subspace topology on $\mathbb{Z}_{\geq 0}$.
Show that the subspace topology on $\mathbb{Z}_{\geq 0}$ (as a subspace of $\mathbb{R}_{\geq 0}$ with the ~~discrete~~ standard topology) is the discrete topology.

~~Remarks~~ Let (X, d) be a metric space,
$d: X \times X \to \mathbb{R}_{\geq 0}$ is the metric.
Let $Y \subseteq X$. Show that $d_Y : Y \times Y \to \mathbb{R}_{\geq 0}$ given by
$$d_Y(y_1, y_2) = d(y_1, y_2)$$
is a metric on Y. This is the "subspace metric".

Product

Let X and Y be sets. The product of X and Y is the set

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}.$$

Let X and Y be metric spaces. The product of X and Y is the metric space

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ with

$d: (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ given by

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} d_X(x_1,x_2) + d_Y(y_1,y_2) & \text{on Mon+Wed} \\ \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\} & \text{on Tues+Thurs} \\ \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}, & \text{on all other days} \end{cases}$$

Exercise Show that all 3 "product metrics" produce the same ~~topology~~ metric space on $X \times Y$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The product of X and Y is the ~~metric sp~~ topological space

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$$

with topology given by

$$\left\{\begin{matrix}\text{open sets} \\ \text{of } X \times Y\end{matrix}\right\} = \left\{\begin{matrix}\text{unions of} \\ U \times V \text{ with } U \text{ open in } X \\ \text{and } V \text{ open in } Y\end{matrix}\right\}.$$

The favorite example: $\mathbb{R}^2$

$$\mathbb{R}^2 = \{(x_1,x_2) \mid x_1, x_2 \in \mathbb{R}\} = \mathbb{R}\times\mathbb{R}$$

is an $\mathbb{R}$-vector space with addition $\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(x_1,x_2)+(y_1,y_2) = (x_1+y_1, x_2+y_2)$$

and scalar multiplication $\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$c(x_1,x_2) = (cx_1, cx_2).$$

$\mathbb{R}^2$ is a pos. def. inner product space with

$$\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_1 + x_2y_2$$

and a normed vector space with

$$\|(x_1,x_2)\| = \sqrt{\langle (x_1,x_2),(x_1,x_2)\rangle} = \sqrt{x_1^2+x_2^2}$$

and a metric space with metric

$$d((x_1,x_2),(y_1,y_2)) = \|(y_1,y_2)-(x_1,x_2)\|$$

$$= \|(y_1-x_1, y_2-x_2)\| = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$$

and a uniform space with uniformity

$$\mathcal{X} = \left\{ V \subseteq \mathbb{R}^2\times\mathbb{R}^2 \;\middle|\; \begin{matrix}\text{there exists } \varepsilon \in \mathbb{R}_{>0} \text{ with} \\ V \supseteq B_\varepsilon\end{matrix} \right\}$$

where $B_\varepsilon = \{((x_1,x_2),(y_1,y_2)) \mid d((x_1,x_2),(y_1,y_2)) < \varepsilon\}$

and a *topological space* with topology

$$\mathcal{T} = \left\{ U \subseteq \mathbb{R}^2 \;\middle|\; \begin{array}{l} U \text{ is a union} \\ \text{of } B_\varepsilon((x_1,x_2)) \end{array} \right\}$$

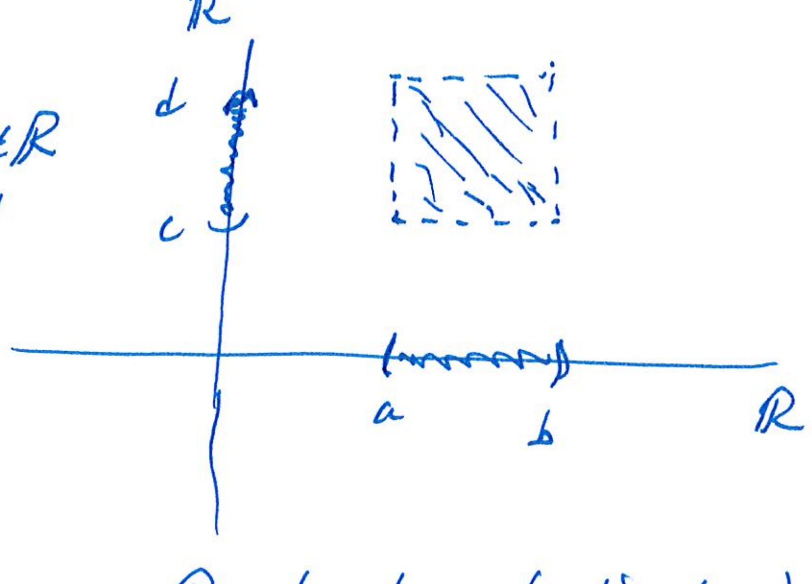
where $B_\varepsilon((x_1,x_2)) = \{ (y_1,y_2) \in \mathbb{R}^2 \mid d((y_1,y_2),(x_1,x_2)) < \varepsilon \}$.

Open balls $B_\varepsilon((x_1,x_2))$

The *product topology* on $\mathbb{R} \times \mathbb{R}$ is

$$\mathcal{T}_{prod} = \left\{ U \subseteq \mathbb{R}^2 \;\middle|\; \begin{array}{l} U \text{ is a union of} \\ (a,b) \times (c,d) \end{array} \right\}$$

$c, d \in \mathbb{R}$
$c < d$



(a,b) $a, b \in \mathbb{R}$
$a < b$

Rectangles $(a,b) \times (c,d) = \left\{ (x_1,x_2) \;\middle|\; \begin{array}{l} a < x_1 < b \\ c < x_2 < d \end{array} \right\}$

Exercise

Show that $\mathcal{T}_{st} = \mathcal{T}_{prod}$