Lecture 8: Metric and Hilbert Spaces

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Cauchy-Schwarz

Let $(V, \langle\, \rangle)$ be a positive definite inner product space. Then

(a) If $x, y \in V$ then $|\langle x,y\rangle| \le \|x\|\cdot\|y\|$.

(b) If $x,y \in V$ then $\|x+y\| \le \|x\| + \|y\|$

Proof

(a) Assume $x, y \in V$. Let $W = \mathrm{span}\{x,y\}$.

Case 0: $\dim(W) = 0$.

Then $x = 0$ and $y = 0$ and

$$|\langle x,y\rangle| = 0 = \|x\|\cdot\|y\|.$$

Case 1: $\dim(W) = 1$.

Then there exists $c \in \mathbb{K}$ with $y = cx$ and

$$|\langle x,y\rangle| = |\langle x,cx\rangle| = |c|\cdot\|x\|^2 = \|x\|\cdot\|cx\| = \|x\|\cdot\|y\|.$$

Case 2: $\dim(W) = 2$

The matrix of $\langle\, ,\, \rangle$ on W is

$$\begin{pmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{pmatrix} = \begin{pmatrix} \|x\|^2 & \langle x,y\rangle \\ \langle y,x\rangle & \|y\|^2 \end{pmatrix}$$

Let

$$w_1 = \frac{x}{\|x\|} \quad \text{and} \quad w_2 = \frac{y - \langle w_1, y\rangle w_1}{\|y - \langle w_1,y\rangle w_1\|}.$$

Then
$$\begin{pmatrix} \langle w_1,w_1\rangle & \langle w_1,w_2\rangle \\ \langle w_2,w_1\rangle & \langle w_2,w_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $x = a_1w_1 + a_2w_2$ and $y = b_1w_1 + b_2w_2$.

Then
$$|\langle x,y\rangle| = |a_1b_1 + a_2b_2|,$$
$$\|x\|^2 = a_1^2 + a_2^2$$
$$\|y\|^2 = b_1^2 + b_2^2.$$

Then, since $0 \le (a_1b_2 - a_2b_1)^2 = -2a_1b_1a_2b_2 + a_1^2b_2^2 + a_2^2b_1^2$,
$$\begin{aligned} |\langle x,y\rangle|^2 = (a_1b_1 + a_2b_2)^2 &= a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 \\ &\le a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_2^2 = (a_1^2+a_2^2)(b_1^2+b_2^2) \\ &= \|x\|^2\cdot\|y\|^2. \end{aligned}$$

So $|\langle x,y\rangle| \le \|x\|\cdot\|y\|$.

(b) Assume $x,y \in V$.

To show: $\|x+y\| \le \|x\| + \|y\|$.

To show: $\|x+y\|^2 \le (\|x\| + \|y\|)^2$.

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle)$$

Then
$$\mathrm{Re}(\langle x,y\rangle) \le \sqrt{\mathrm{Re}(\langle x,y\rangle)^2} \le \sqrt{\mathrm{Re}(\langle x,y\rangle)^2 + \mathrm{Im}(\langle x,y\rangle)^2}$$
$$= |\langle x,y\rangle| \le \|x\|\cdot\|y\|, \text{ by Cauchy-Schwarz.}$$

So $\|x+y\|^2 \le \|x\|^2 + \|y\|^2 + 2\|x\|\cdot\|y\| = (\|x\| + \|y\|)^2$. //

Hölder and Minkowski inequalities

Theorem Let $q \in \mathbb{R}_{>1}$ and $p \in \mathbb{R}_{>1}$, with $\frac{1}{p}+\frac{1}{q}=1$.

Let $x = (x_1, x_2, x_3, \ldots) \in \ell^p$,

$y = (y_1, y_2, \ldots) \in \ell^q$,

and let $\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + \cdots$.

Then $|\langle x,y\rangle| \le \|x\|_p \cdot \|y\|_q$ and

$\|x+y\|_p \le \|x\|_p + \|y\|_p$.

Proof To show: (a) $|\langle x,y\rangle| \le \|x\|_p \cdot \|y\|_p$

(b) $\|x+y\|_p \le \|x\|_p + \|y\|_p$

(b) $$\|x+y\|_p^p = \sum_{i=1}^n |x_i+y_i|^p$$

$$= \sum_{i=1}^n |x_i+y_i|\,|x_i+y_i|^{p-1}$$

$$\le \sum_{i=1}^n (|x_i|+|y_i|)\,|x_i+y_i|^{\frac{p}{q}},$$ since $|x_i+y_i| \le |x_i|+|y_i|$ and $p-1 = p(1-\frac{1}{p}) = p\cdot\frac{1}{q}$.

$$= \sum_{i=1}^n |x_i|\,|x_i+y_i|^{\frac{p}{q}} + \sum_{i=1}^n |y_i|\,|x_i+y_i|^{p/q}$$

$$\le \|x\|_p \cdot \|(|x_1+y_1|^{p/q}, |x_2+y_2|^{p/q}, \ldots)\|_q$$

$$+ \|y\|_p \cdot \|(|x_1+y_1|^{p/q}, |x_2+y_2|^{p/q}, \ldots)\|_q.$$

$$= (\|x\|_p + \|y\|_p) \cdot \|x+y\|_p^{p/q},$$

since

$$\|(|x_1+y_1|^{p/q}, |x_2+y_2|^{p/q}, \dots)\|_q = \left(\sum_{i=1}^n (|x_i+y_i|^{p/q})^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{i=1}^n |x_i+y_i|^p\right)^{\frac{1}{p}\cdot\frac{p}{q}} = (\|x+y\|_p)^{p/q}.$$

Thus:

$$\|x+y\|_p^p \leq (\|x\|_p + \|y\|_p)\, \|x+y\|_p^{p-1}$$

So

$$\|x+y\|_p = \frac{\|x+y\|_p^p}{\|x+y\|_p^{p-1}} \leq (\|x\|_p + \|y\|_p) \frac{\|x+y\|_p^{p-1}}{\|x+y\|_p^{p-1}}$$

$$= (\|x\|_p + \|y\|_p).$$