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Lecture 7 Metric and Hilbert spaces

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Function spaces

$$\mathbb{R}^n = \{ x : \{1, 2, \ldots, n\} \to \mathbb{R} \}$$

$$\mathbb{R}^\infty = \{ x : \mathbb{Z}_{>0} \to \mathbb{R} \} \text{ and}$$

$$\ell^p = \{ (x_1, x_2, \ldots) \mid x_j \in \mathbb{R} \text{ and } \|x\|_p < \infty \}$$

where $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$.

Bounded continuous functions

Let (X, d) and (Y, ρ) be metric spaces.

Let

$$BC(X, Y) = \left\{ f : X \to Y \;\middle|\; \begin{array}{l} f \text{ is continuous and} \\ f \text{ is bounded} \end{array} \right\}$$

with metric $d_\infty : BC(X,Y) \times BC(X,Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f, g) = \sup \{ \rho(f(x), g(x)) \mid x \in X \}.$$

A function $f : X \to Y$ is <u>bounded</u> if $f(X)$ is bounded.

A subset $E \subseteq Y$ is <u>bounded</u> if there exists $y \in Y$ and $M \in \mathbb{R}_{\geq 0}$ such that

if $x \in E$ then $d(x, y) < M$.

Bounded linear operators

Let $(V, \| \|_V)$ and $(W, \| \|_W)$ be normed vector spaces.

The space of bounded linear transformations from V to W is

$$B(V,W) = \left\{ T: V \to W \;\middle|\; \begin{array}{l} T \text{ is a linear transformation} \\ \|T\| < \infty \end{array} \right\}$$

where

$$\|T\| = \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \;\middle|\; v \in V \right\}$$

A linear transformation $T: V \to W$ is <u>bounded</u> if $\|T\| < \infty$.

<u>Theorem</u> Exercise 5.6.3(4) in the notes.
Let V and W be normed vector spaces and let $T: V \to W$ be a linear transformation.
The following are equivalent

(a) $\|T\| < \infty$

(b) $T: V \to W$ is uniformly continuous

(c) $T: V \to W$ is continuous.

Rubinstein Example 21.7

$$BC([a,b],\mathbb{C}) = \left\{ x:[a,b]\to\mathbb{C} \;\middle|\; \begin{array}{l} x \text{ is continuous} \\ \|x\|_\infty < \infty \end{array} \right\}$$

where $\|x\|_\infty = \sup\{|x(j)| \mid j\in[a,b]\}$.

Let $k:[a,b]\times[a,b]\to\mathbb{C}$ be a continuous function

Define a linear transformation

$$T: BC([a,b],\mathbb{C}) \to BC([a,b],\mathbb{C})$$

by

$$(Tx)(t) = \int_a^b k(t,s)\,x(s)\,ds$$

Matrices:

$$V = \mathbb{C}^n = \{x:\{1,2,\dots,n\}\to\mathbb{C}\}$$

Let $\begin{array}{l} k:\{1,\dots,n\}\times\{1,\dots,n\}\to\mathbb{C} \\ \qquad (i,j)\longmapsto k(i,j)\end{array}$ be an $n\times n$ matrix.

Define a linear transformation $T: V\to V$ by

$$(Tx)(i) = \sum_{j=1}^{n} k(i,j)\,x(j)$$

Exercise 3.5.2 (1) Let (X,d) and (Y,ρ) be metric spaces and let $f: X \to Y$ be a function.
The function $f: X \to Y$ is continuous if and only if f satisfies
if $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.

Exercise 3.5.2 (2) Let (X,d) and (Y,ρ) be metric spaces and let $f: X \to Y$ be a function
The function $f: X \to Y$ is uniformly continuous if and only if f satisfies
if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.