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Lecture 6 Metric and Hilbert spaces (1)

Univ. of Melbourne

Function spaces:

(1) $\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$

$= \{x : \{1, 2, \ldots, n\} \to \mathbb{R}\}$

$= \{\text{functions from } \{1, 2, \ldots n\} \text{ to } \mathbb{R}\}$

The function

$$x : \{1, 2, \ldots, n\} \to \mathbb{R}$$

$$\begin{aligned} 1 &\mapsto x_1 \\ 2 &\mapsto x_2 \\ &\vdots \\ n &\mapsto x_n \end{aligned}$$

corresponds to $(x_1, x_2, \ldots, x_n)$

the sequence.

Possible norms on $\mathbb{R}^n$

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \quad \text{(standard norm)}$$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$$

$$\|x\|_\infty = \sup\{|x_1|, |x_2|, \ldots, |x_n|\}$$

(2) $\mathbb{R}^\infty = \{ x = (x_1, x_2, x_3, \ldots) \mid x_i \in \mathbb{R} \}$

$= \{$sequences $(x_1, x_2, \ldots)$ in $\mathbb{R}\}$

$= \{ x : \mathbb{Z}_{>0} \to \mathbb{R} \}$

$= \{$functions from $\mathbb{Z}_{>0}$ to $\mathbb{R}\}$.

The sequence

$(x_1, x_2, x_3, \ldots)$ in $\mathbb{R}$

corresponds to the function

$$\begin{aligned} x : \mathbb{Z}_{>0} &\longrightarrow \mathbb{R} \\ 1 &\longmapsto x_1 \\ 2 &\longmapsto x_2 \\ &\vdots \end{aligned}$$

<u>Possible norms on $\mathbb{R}^\infty$</u>

$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$ gives ℓ^2

$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$ gives ℓ^p

$\|x\|_\infty = \sup\{ |x_1|, |x_2|, \ldots \}$ gives ℓ^∞.

More function spaces:

(3) $F = \{\text{functions } f: [0,1] \to \mathbb{R}\}$ or

$F = \{\text{functions } f: X \to \mathbb{R}\}$.

with
$$\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}.$$

(4) Let $(V, \| \|)$ and $(W, \| \|)$ be normed vector spaces. Let

$F = \{\text{linear transformations } T: V \to W\}$.

with
$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \;\middle|\; v \in V\right\}$$

(5) Let (X, d) and (Y, ρ) be metric spaces.

Let $F = \{\text{functions } f: X \to Y\}$

with
$d_\infty: F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ given by
$$d_\infty(f_1, f_2) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

<u>Remark on (4)</u> If $(V, \| \|)$ and $(W, \| \|)$ are normed vector spaces then

$\mathcal{B}(V, W) = \{\text{linear transformations } T: V \to W \mid \|T\| < \infty\}$

is a normed vector space.

"Polynomial" or "Power series" number systems.

$$\mathbb{R}[t] = \left\{ a_0 + a_1 t + a_2 t^2 + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{R} \\ \text{all but a fin.} \\ \text{number of } a_i \text{ are } 0 \end{array} \right\}$$

$$\mathbb{R}[[t]] = \{ a_0 + a_1 t + a_2 t^2 + \cdots \mid a_j \in \mathbb{R} \}$$

Examples: $3 + 4t + 5t^2 \in \mathbb{R}[t]$.

$$e^t = 1 + \tfrac{1}{1!} t + \tfrac{1}{2!} t^2 + \tfrac{1}{3!} t^3 + \cdots \quad \in \mathbb{R}[[t]]$$

$$\sin t = t - \tfrac{1}{3!} t^3 + \tfrac{1}{5!} t^5 + \cdots \quad \in \mathbb{R}[[t]].$$

$$\cos t = 1 - \tfrac{1}{2!} t^2 + \tfrac{1}{4!} t^4 - \cdots \quad \in \mathbb{R}[[t]].$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \quad \in \mathbb{R}[[t]].$$

$$\log(1+t) = t - \tfrac{1}{2} t^2 + \tfrac{1}{3} t^3 - \tfrac{1}{4} t^4 + \cdots \quad \in \mathbb{R}[[t]]$$

Define

$$|a_\ell t^\ell + a_{\ell+1} t^{\ell+1} + \cdots| = e^{-\ell}.$$

Then $\mathbb{R}[[t]]$ with $d : \mathbb{R}[[t]] \times \mathbb{R}[[t]] \to \mathbb{R}_{\geq 0}$ given by

$$|x - y| = d(x,y) \quad \text{is a metric space.}$$

In

$$\mathbb{Q}_7 = \left\{ a_\ell 7^\ell + a_{\ell+1} 7^{\ell+1} + \cdots \;\middle|\; \begin{array}{l} a_j \in \{0,1,2,3,4,5,6\} \\ \ell \in \mathbb{Z} \end{array} \right\}$$

Then

$$888 = 6 + 0\cdot 7 + 4\cdot 7^2 + 1\cdot 7^3 + 0\cdot 7^4 + 0\cdot 7^5 + \cdots$$

$$\text{"="}\ .60410000\ldots$$

$$\frac{-1}{6} = \frac{1}{1-7} = 1 + 7 + 7^2 + 7^3 + \cdots$$

$$\text{"="}\ 1.1111111\ldots$$

$$-1 = 6\cdot\left(\frac{-1}{6}\right) = 6 + 6\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + \cdots$$

$$\text{"="}\ 6.6666666\ldots$$

In

$$\mathbb{R}_{10} = \left\{ a_\ell \left(\tfrac{1}{10}\right)^\ell + a_{\ell+1}\left(\tfrac{1}{10}\right)^{\ell+1} + \cdots \;\middle|\; \begin{array}{l} a_j \in \{0,1,2,\ldots 9\} \\ \ell \in \mathbb{Z} \end{array} \right\}$$

Then

$$1.11111\ldots = 1 + 1\cdot\frac{1}{10} + 1\cdot\left(\frac{1}{10}\right)^2 + \cdots$$

$$= \frac{1}{1-\frac{1}{10}} = \frac{10}{9}.$$

$$6.6666\ldots = 6 + 6\cdot\frac{1}{10} + 6\cdot\left(\frac{1}{10}\right)^2 + \cdots$$

$$= 6\cdot\left(1 + 1\cdot\frac{1}{10} + 1\cdot\left(\frac{1}{10}\right)^2 + \cdots\right)$$

$$= 6\cdot\frac{10}{9} = \frac{2\cdot 10}{3} = \frac{20}{3} = 6\tfrac{2}{3}.$$