Metric and Hilbert Spaces. Lecture 3 28.07.2016 ①
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A <u>metric space</u> is a set X with a function

$$d: X \times X \to \mathbb{R}_{\geq 0} \quad \text{such that}$$

(a) If $x \in X$ then $d(x,x) = 0$

(b) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$.

(c) If $x, y \in X$ then $d(x,y) = d(y,x)$

(d) If $x, y, z \in X$ then $d(x,y) \leq d(x,z) + d(z,y)$.

Let (X,d) be a metric space.

Let $\varepsilon \in \mathbb{R}_{>0}$. The <u>$\varepsilon$-diagonal</u> is

$$B_\varepsilon = \{ (x,y) \in X \times X \mid d(x,y) < \varepsilon \}$$

Let $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$. The <u>ε-ball centered at x</u> is

$$B_\varepsilon(x) = \{ y \in X \mid d(x,y) < \varepsilon \}.$$

The <u>metric space uniformity</u> $\mathcal{E}$ is the collection of $E \subseteq X \times X$ such that

E contains an ε-diagonal.

The <u>metric space topology</u> $\mathcal{T}$ is the collection of $U \subseteq X$ such that

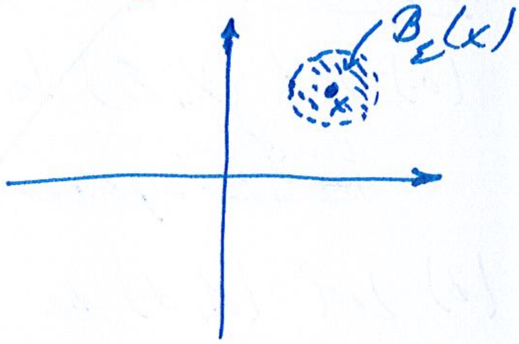
U is a union of ε-balls.

<u>Favorite example</u>: $X = \mathbb{R}\times\mathbb{R}$ with $d : X\times X \to \mathbb{R}_{\geq 0}$

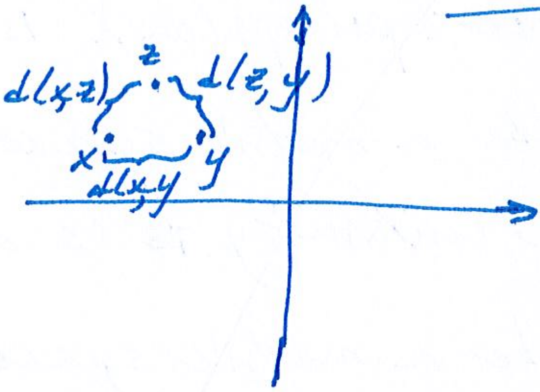
given by $d((x_1,x_2),(y_1,y_2)) = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$

Let $x=(x_1,x_2)\in\mathbb{R}^2$ and $\varepsilon\in\mathbb{R}_{>0}$. Then

$B_\varepsilon(x) = \{y\in\mathbb{R}^2 \mid d(x,y)<\varepsilon\}$



The triangle inequality

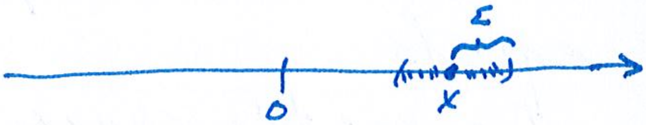


<u>Favorite example</u>: $X=\mathbb{R}$ with $d : X\times X \to \mathbb{R}_{\geq 0}$

given by $d(x,y) = |y-x| = \sqrt{(y-x)^2}$

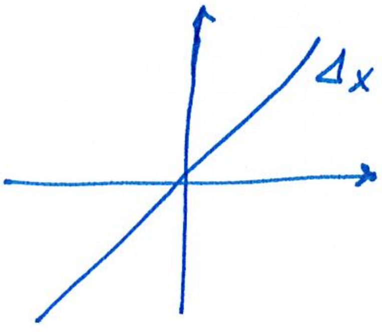
Let $x\in\mathbb{R}$ and $\varepsilon\in\mathbb{R}_{>0}$. Then

$B_\varepsilon(x) = \{y\in\mathbb{R} \mid d(y,x)<\varepsilon\}$

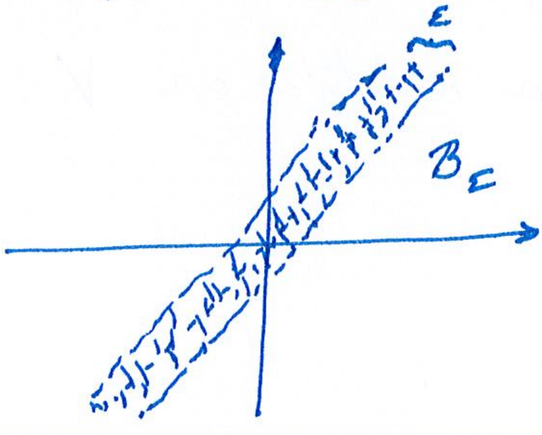


Let $\varepsilon\in\mathbb{R}_{>0}$. Then

$$B_\varepsilon = \{(x,y) \mid d(x,y)<\varepsilon\} \subseteq \mathbb{R}^2$$



and



Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

A <u>vector space over $\mathbb{K}$</u> is a set V with functions

$$\begin{matrix} V\times V \to V \\ (v_1,v_2)\mapsto v_1+v_2 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{K}\times V \to V \\ (c,v)\mapsto cv \end{matrix}$$

such that

(a) If $v_1, v_2, v_3 \in V$ then $v_1+(v_2+v_3) = (v_1+v_2)+v_3$,

(b) If $v_1, v_2 \in V$ then $v_1+v_2 = v_2+v_1$,

(c) There exists $0\in V$ such that
if $v\in V$ then $0+v=v$,

(d) If $v\in V$ then there exists $-v\in V$ such that
$v+(-v)=0$,

(e) If $c_1,c_2\in\mathbb{K}$ and $v\in V$ then $c_1(c_2v)=(c_1c_2)v$,

(f) If $c_1,c_2\in\mathbb{K}$ and $v\in V$ then $(c_1+c_2)v = c_1v+c_2v$,

(g) If $c\in\mathbb{K}$ and $v_1,v_2\in V$ then $c(v_1+v_2) = cv_1+cv_2$,

(h) If $v\in V$ then $1\cdot v=v$.

A <u>normed vector space</u> is a vector space V with a function $\begin{matrix} V\to\mathbb{R}_{\geq 0} \\ v\mapsto \|v\| \end{matrix}$ such that

(a) If $c\in\mathbb{K}$ and $v\in V$ then $\|cv\| = |c|\,\|v\|$,

(b) If $v\in V$ and $\|v\|=0$ then $v=0$,

(c) If $v_1,v_2\in V$ then $\|v_1+v_2\| \leq \|v_1\|+\|v_2\|$.

Let $(V, \|\cdot\|)$ be a normed vector space. A. Ram

The <u>normed vector space metric</u> is

$d: V\times V \to \mathbb{R}_{\geq 0}$ given by $d(x,y) = \|y-x\|$.

<u>Theorem</u>

(a) Let $(X, \mathcal{E})$ be a uniform space. The uniform space topology on X is a topology on X.

(b) Let (X,d) be a metric space. The metric space uniformity is a uniformity on X.

(c) Let (X,d) be a metric space. The metric space topology is a topology on X.

(d) Let (X,d) be a metric space. Let $\mathcal{E}$ be the metric space uniformity on X. Then the uniform space topology for $(X,\mathcal{E})$ equals the metric space topology for (X,d).

(e) Let $(V, \|\cdot\|)$ be a normed vector space. The normed vector space metric is a metric on V.

Favorite examples

(1) Let $X=\{1,2\}$ and let $\mathcal{T}=\{\emptyset,\{1\},X\}$.
Then $\mathcal{T}$ is a topology on X and there is no uniformity $\mathcal{E}$ on X such that
$\mathcal{T}$ is the uniform space topology of $(X,\mathcal{E})$.

(2) Let $X=\mathbb{R}$ and define $d: X\times X\to\mathbb{R}_{\geq 0}$ by

$$d(x,y)=\begin{cases}0, & \text{if } x=y,\\ 1, & \text{if } x\neq y.\end{cases}$$

Then d is a metric on $\mathbb{R}$ and the metric space topology of (X,d) is the discrete topology.

(3) Let $X=\mathbb{R}$ and let

$$\mathcal{T}=\{E\subseteq X \mid E \text{ is infinite}\}$$

Then $\mathcal{T}$ is a topology on X.

(a) Is there a uniformity $\mathcal{E}$ on X such that $\mathcal{T}$ is the uniform space topology of $(X,\mathcal{E})$?

(b) Show that there is no metric $d: X\times X\to\mathbb{R}_{\geq 0}$ such that $\mathcal{T}$ is the metric space topology of (X,d).