Lecture 35: Metric and Hilbert Spaces

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Review of limits

Definitions from 2nd year Real Analysis:

(1) Let (X,d) be a metric space and let $f\colon X\to\mathbb{R}$.

$\lim_{x\to a} f(x) = l$ means

if $\varepsilon\in\mathbb{R}_{>0}$ then there exists $\delta\in\mathbb{R}_{>0}$ such that

if $d(x,a)<\delta$ then $d(f(x),l)<\varepsilon$.

(2) Let (X,d) be a metric space and let $(a_1,a_2,\ldots)$ be a sequence in X. Let $l\in X$.

$\lim_{n\to\infty} a_n = l$ means

if $\varepsilon\in\mathbb{R}_{>0}$ then there exists $N\in\mathbb{Z}_{>0}$ such that

if $n\in\mathbb{Z}_{\geq N}$ then $d(a_n,l)<\varepsilon$.

Definitions from 3rd year Metric and Hilbert spaces:

(1) Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be topological spaces. Let $f\colon X\to Y$ be a function and let $a\in X$ and $y\in Y$.

$\lim_{x\to a} f(x) = y$ means

if $N\in\mathcal{N}(y)$ then there exists $P\in\mathcal{N}(a)$ such that

$N\supseteq f(P)$.

(2) Let $(X, \mathcal{T})$ be a topological space. Let $(x_1, x_2, \ldots)$ be a sequence in X. Let $z \in X$.

$$\lim_{n\to\infty} x_n = z \quad \text{means}$$

if $N \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.

<u>Review of neighborhoods</u>

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$.
A <u>neighborhood of x</u> is a subset N of X such that
there exists $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq N$.

The <u>neighborhood filter of x</u> is

$$\mathcal{N}(x) = \{\text{neighborhoods of } x\}.$$

<u>Neighborhoods in metric spaces</u>

Let (X, d) be a metric space.
Let $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$. The <u>open ball of radius ε at x</u> is

$$B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}.$$

Let $x \in X$. A <u>neighborhood of x</u> is a subset N of X such that

there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(x) \subseteq N$.

The <u>neighborhood filter of x</u> is

$$\mathcal{N}(x) = \left\{ N \subseteq X \ \middle|\ \begin{matrix}\text{there exists } \varepsilon \in \mathbb{R}_{>0} \\ \text{such that } B_\varepsilon(x) \subseteq N\end{matrix} \right\}$$

<u>Review of Cauchy sequences and uniform continuity.</u>

Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces.
A <u>uniformly continuous function from X to Y</u> is a function $f: X \to Y$ such that

if $V \in \mathcal{X}_Y$ then $(f \times f)^{-1}(V) \in \mathcal{X}_X$.

Let $(X, \mathcal{X})$ be a uniform space.
A <u>Cauchy sequence in X</u> is a sequence $(x_1, x_2, \ldots)$ in X such that

if $V \in \mathcal{X}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $m, n \in \mathbb{Z}_{\geq N}$ then $(x_m, x_n) \in V$.

Cauchy sequences and uniform continuity in metric spaces

Let (X, d) be a metric space.

Let $\varepsilon \in \mathbb{R}_{>0}$. The diagonal of width ε, or ε-diagonal, is

$$B_\varepsilon = \{(y,x) \in X \times X \mid d(x,y) < \varepsilon\}.$$

Let (X, d_X) and (Y, d_Y) be metric spaces.

A uniformly continuous function from X to Y is a function $f: X \to Y$ such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x, y \in X$ and $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

Let (X, d) be a metric space.

A Cauchy sequence on X is a sequence $(x_1, x_2, \ldots)$ in X such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $m, n \in \mathbb{Z}_{\geq N}$ then $d(x_m, x_n) < \varepsilon$

Let (X, d) be a metric space. The metric space uniformity is

$$\mathcal{X} = \left\{ V \subseteq X \times X \;\middle|\; \begin{array}{l} \text{there exists } \varepsilon \in \mathbb{R}_{>0} \\ \text{such that } D \supseteq B_\varepsilon \end{array} \right\}$$