Lecture 34: Metric and Hilbert Spaces

18.10.2016
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Review of bounded operators, continuity, and uniform continuity

Proposition Let $(V, \| \|)$ and $(W, \| \|)$ be normed vector spaces. Let $T: V \to W$ be a linear operator. The following are equivalent:

(a) T is bounded

(b) T is continuous

(c) T is uniformly continuous.

Proof (a) $\Rightarrow$ (c). Assume $T \in B(V, W)$.

To show: T is uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in V$ and $d(x,y) < \delta$ then $d(Tx, Ty) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in V$ ~~then~~ and $d(x,y) < \delta$ then $d(Tx, Ty) < \varepsilon$.

Assume $x, y \in V$ and $d(x,y) < \delta$.

To show: $d(Tx, Ty) < \varepsilon$.

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$$d(Tx,Ty) = \|Tx - Ty\| = \|T(x-y)\|$$
$$\leq \|T\| \cdot \|x-y\| = \|T\|\, d(x,y)$$
$$< \|T\| \cdot \delta < \varepsilon.$$

So T is uniformly continuous.

(b) $\Rightarrow$ (a): To show: If T is continuous then T is bounded.

Assume T is continuous.

To show: $\|T\| < \infty$.

To show: There exists $C \in \mathbb{R}_{>0}$ such that if $u \in V$ then $\|Tu\| \leq C\|u\|$.

Since T is continuous, T is continuous at 0.

So there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in V$ and $\|x\| < \delta$ then $\|Tx\| < 1$.

Let
$$C = \frac{2}{\delta}$$

To show: If $u \in V$ then $\|Tu\| \leq C\|u\|$.

Assume $u \in V$.

To show: $\|Tu\| \leq C\|u\|$.

Let
$$x = \frac{\delta}{2} \cdot \frac{u}{\|u\|} \quad \text{so that} \quad \|x\| < \frac{\delta}{2}.$$

Then
$$1 > \|Tx\| = \left\|T\left(\frac{\delta}{2}\cdot\frac{u}{\|u\|}\right)\right\| = \frac{\delta}{2\|u\|}\|Tu\|$$

So
$$\|Tu\| < \frac{2}{\delta}\|u\| = c\|u\|.$$

So T is bounded.

For (c) $\Rightarrow$ (b) see below. //.

<u>Proposition</u> Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces and let $\mathcal{T}_X$ be the uniform space topology on $(X, \mathcal{X}_X)$ and let $\mathcal{T}_Y$ be the uniform space topology on $(Y, \mathcal{X}_Y)$. Let $f: X \to Y$ be a uniformly continuous function. Then $f: X \to Y$ is continuous.

<u>Proof</u>: Assume $f: X \to Y$ is uniformly continuous.

To show: $f: X \to Y$ is continuous.

To show: If $a \in A$ then $f: X \to Y$ is continuous at a.

Assume $a \in X$.

To show: f is continuous at a.

To show: If $V \in N(f(a))$ then $f^{-1}(V) \in N(a)$

Assume $V \in N(f(a))$.

To show: $f^{-1}(V) \in \mathcal{N}(a)$

To show: There exists $D \in \mathcal{X}$ such that
$f^{-1}(V) \supseteq B_D(a)$.

Since $V \in \mathcal{N}(f(a))$ there exists $C \in \mathcal{X}_Y$ such that $V \supseteq B_C(f(a))$.

Let $D = (f \times f)^{-1}(C)$.

To show: $f^{-1}(V) \supseteq B_D(a)$

To show: If $y \in B_D(a)$ then $y \in f^{-1}(V)$.

Assume $y \in B_D(a)$.

Then $(a,y) \in D$

So $(a,y) \in (f \times f^{-1})(C)$

So $(f(a), f(y)) \in C$.

So $f(y) \in B_C(f(a))$

So $f(y) \in V$.

So $y \in f^{-1}(V)$.

So $f^{-1}(V) \supseteq B_D(a)$

So $f^{-1}(V) \in \mathcal{N}(a)$

So f is continuous at a.

So f is continuous. //.