Lecture 33: Metric and Hilbert Spaces

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Examples of duals

(a) Let $p \in \mathbb{R}_{>1}$ and $q \in \mathbb{R}_{>1}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

$$(\ell^p)^* = \ell^q.$$

(b) $(\ell^1)^* = \ell^\infty$

(c) $(\ell^\infty)^* \neq \ell^1$

Examples of linear operators

A <u>bounded linear operator from V to W</u> is a linear operator $T: V \to W$ such that there exists $c \in \mathbb{R}_{>0}$ such that if $u \in V$ then $\|Tu\| \le c\|u\|$.

The <u>norm of T</u> is the minimal c that works.

Examples

(1) Let V be a normed vector space and

$$\begin{matrix} 1: V \to V \\ x \mapsto x \end{matrix} \quad \text{and} \quad \begin{matrix} 0: V \to V \\ x \mapsto 0 \end{matrix}$$

Then $\|1\| = 1$ and $\|0\| = 0$.

(2) Let $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$.
with the sup norm

$$\|f\| = \sup \{ |f(t)| \mid t \in [a,b] \}$$

Let $T : C[a,b] \to \mathbb{R}$ be given by

$$Tf = \int_a^b f(t)\,dt$$

If $x : [a,b] \to \mathbb{R}$ is the function given by $x(t) = 1$
then

$$Tx = \int_a^b dt = b-a \text{ and } \|Tx\| = b-a = (b-a)\|x\|$$

since $\|x\| = 1$. So $\|T\| \geq b-a$.

If $f \in C[a,b]$ then

$$\|Tf\| = \left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt \leq \|f\|(b-a).$$

So $\|T\| \leq b-a$. Thus $\|T\| = b-a$.

(3) Integral operators. Let

$$C[a,b] = \{ f : [a,b] \to \mathbb{C} \mid f \text{ is continuous} \}$$

with norm given by

$$\|f\| = \sup \{ |f(t)| \mid t \in [a,b] \}$$

Let $K : [a,b] \times [a,b] \to \mathbb{C}$ be a continuous function.

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Define $T: C[a,b] \to C[a,b]$ by

$$(Tf)(t) = \int_a^b K(t,s) f(s)\, ds.$$

(generalised matrix multiplication!).

To show: (a) If $f \in C[a,b]$ then $Tf \in C[a,b]$.

(b) T is a bounded linear operator.

(a) Assume $f \in C[a,b]$.

To show: Tf is continuous.

In fact we will show: Tf is uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $t, t' \in [a,b]$ and $|t-t'| < \delta$ then

$$|Tf(t) - Tf(t')| < \varepsilon.$$

Assume $\varepsilon \in \mathbb{R}_{>0}$

Since $[a,b] \times [a,b]$ is compact and K is continuous then K is uniformly continuous.

Let $\delta \in \mathbb{R}_{>0}$ be such that if $s, s', t, t' \in [a,b]$ and

$$d((s,t),(s',t')) < \delta \text{ then } |K(s,t) - K(s',t')| < \frac{\varepsilon}{(b-a)\|f\|}.$$

To show: If $t, t' \in [a,b]$ and $|t-t'| < \delta$ then

$$|Tf(t) - Tf(t')| < \varepsilon$$

Assume $t, t' \in [a,b]$ and $|t-t'| < \delta$.

To show: $|Tf(t) - Tf(t')| < \varepsilon$.

$$|Tf(t) - Tf(t')| = \left| \int_a^b (K(s,t) - K(s,t')) f(s)\, ds \right|$$

$$\leq \int_a^b |K(s,t) - K(s,t')| \cdot |f(s)|\, ds$$

$$< \frac{\varepsilon}{(b-a)\|f\|} \cdot (b-a)\|f\| = \varepsilon.$$

So Tf is uniformly continuous.

So Tf is continuous and $Tf \in C[a,b]$.

(b) To show: T is a bounded linear operator.

To show: There exists $C \in \mathbb{R}_{>0}$ such that if $f \in C[a,b]$ then $\|Tf\| \leq C\|f\|$.

Since $[a,b] \times [a,b]$ is compact and K is continuous

$K([a,b] \times [a,b])$ is compact and

$K([a,b] \times [a,b])$ is bounded.

Let

$$C = \sup\{ |K(s,t)| \mid s,t \in [a,b] \}$$

To show: If $f \in C[a,b]$ then $\|Tf\| \leq C\|f\|$.

Assume $f \in C[a,b]$.

Then

$$|Tf(t)| \leq \int_a^b |K(s,t)| \cdot |f(s)| \, ds \leq (b-a) C \|f\|.$$

Then

$$\|Tf\| = \sup\{ |Tf(t)| \mid t \in [a,b] \} \leq (b-a) C \|f\|.$$

So $\|T\| \leq (b-a) C$.

So $\|T\|$ is bounded. //.