Lecture 32: Metric and Hilbert Spaces

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Let $p \in \mathbb{R}_{>1}$ and let $q \in \mathbb{R}_{>1}$ be given by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Recall that

$$\ell^p = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|(x_1, x_2, \ldots)\|_p < \infty\}$$

where

$$\|(x_1, x_2, \ldots)\|_p = \Big(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p\Big)^{1/p}.$$

Theorem $(\ell^p)^* = \ell^q$.

Recall that $(\ell^p)^* = B(\ell^p, \mathbb{R})$.

Define

$$\begin{aligned} \Phi : \ell^q &\longrightarrow B(\ell^p, \mathbb{R}) \\ y &\longmapsto \Phi_y : \ell^p \longrightarrow \mathbb{R} \\ & \qquad\qquad x \longmapsto \langle y, x\rangle. \end{aligned}$$

where

$$\langle y, x\rangle = \sum_{i \in \mathbb{Z}_{>0}} y_i x_i, \quad \text{if } y = (y_1, y_2, \ldots) \text{ and } x = (x_1, x_2, \ldots).$$

To show: (a) Φ is a linear transformation

(b) Φ is invertible

(c) If $y \in \ell^q$ then $\|\Phi_y\| = \|y\|$.

(a) To show: (aa) If $y_1, y_2 \in \ell^q$ then $\Phi_{y_1+y_2} = \Phi_{y_1} + \Phi_{y_2}$.

(ab) If $y \in \ell^q$ and $c \in \mathbb{R}$ then $\Phi_{cy} = c\Phi_y$.

(aa) Assume $y_1, y_2 \in \ell^q$.

To show: $\Phi_{y_1+y_2} = \Phi_{y_1} + \Phi_{y_2}$

To show: If $x \in \ell^p$ then $\Phi_{y_1+y_2}(x) = (\Phi_{y_1} + \Phi_{y_2})(x)$.

Assume $x \in \ell^p$.

To show: $\Phi_{y_1+y_2}(x) = (\Phi_{y_1} + \Phi_{y_2})(x)$

$$\Phi_{y_1+y_2}(x) = \langle y_1+y_2, x\rangle = \langle y_1, x\rangle + \langle y_2, x\rangle$$
$$= \Phi_{y_1}(x) + \Phi_{y_2}(x) = (\Phi_{y_1} + \Phi_{y_2})(x).$$

(ab) Assume $y \in \ell^q$ and $c \in \mathbb{R}$.

To show: $\Phi_{cy} = c\Phi_y$.

To show: If $x \in \ell^p$ then $\Phi_{cy}(x) = (c\Phi_y)(x)$.

Assume $x \in \ell^p$.

To show: $\Phi_{cy}(x) = (c\Phi_y)(x)$

$$\Phi_{cy}(x) = \langle cy, x\rangle = c\langle y, x\rangle = c(\Phi_y(x)) = (c\Phi_y)(x).$$

So $\Phi : \ell^q \to \mathcal{B}(\ell^p, \mathbb{R})$ is a linear transformation.

(b) To show: $\Phi: \ell^q \to B(\ell^p, \mathbb{R})$ is invertible.

To show: There exists $\Psi: B(\ell^p, \mathbb{R}) \to \ell^q$ such that $\Phi \circ \Psi = id$ and $\Psi \circ \Phi = id$.

Let $\Psi: B(\ell^p, \mathbb{R}) \to \ell^q$ be given by

$$\Psi(\gamma) = (\gamma(e_1), \gamma(e_2), \ldots)$$

where $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ with 1 in the i^{th} spot.

To show: (ba) $\Phi \circ \Psi = id$.

(bb) $\Psi \circ \Phi = id$.

(ba) To show: If $\gamma \in B(\ell^p, \mathbb{R})$ then $\Phi(\Psi(\gamma)) = \gamma$.

Assume $\gamma \in B(\ell^p, \mathbb{R})$

To show: $\Phi(\Psi(\gamma)) = \gamma$.

To show: If $x \in \ell^p$ then $\Phi(\Psi(\gamma))(x) = \gamma(x)$.

Assume $x \in \ell^p$. Let $x = (x_1, x_2, \ldots)$

To show: $\Phi(\Psi(\gamma))(x) = \gamma(x)$

$$\Phi(\Psi(\gamma))(x) = \Phi(\gamma(e_1), \gamma(e_2), \ldots)(x)$$

$$= \langle(\gamma(e_1), \gamma(e_2), \ldots), (x_1, x_2, \ldots)\rangle = \sum_{i \in \mathbb{Z}_{>0}} \gamma(e_i) x_i$$

$$= \gamma\Big(\sum_{i \in \mathbb{Z}_{>0}} x_i e_i\Big) = \gamma(x).$$

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(bb) To show: $\Psi \circ \Phi = id.$

To show: If $y \in \ell^q$ then $\Psi(\Phi(y)) = y$.

Assume $y \in \ell^q$. Let $y = (y_1, y_2, \ldots)$.

To show: $\Psi(\Phi(y)) = y$.

~~Assume~~ $\Psi(\Phi(y)) = \Psi(\Phi_y) = (\Phi_y(e_1), \Phi_y(e_2), \ldots) = (y_1, y_2, \ldots)$,

since $\Phi_y(e_i) = \langle y, e_i \rangle = \langle (y_1, y_2, \ldots), (0, 0, \ldots, 0, 1, 0, \ldots) \rangle = y_i$.

So $\Psi(\Phi(y)) = y$.

(c) To show: If $y \in \ell^q$ then $\|\Phi_y\| = \|y\|_q$.

Assume $y \in \ell^q$. Let $y = (y_1, y_2, \ldots)$.

To show: (ca) $\|\Phi_y\| \le \|y\|_q$

(cb) $\|\Phi_y\| \ge \|y\|_q$.

(ca) To show: If $x \in \ell^p$ then $|\Phi_y(x)| \le \|x\|_p \|y\|_q$.

Assume $x \in \ell^p$. Let $x = (x_1, x_2, \ldots)$

Then
$$|\Phi_y(x)| = \Big|\sum_{n \in \mathbb{Z}_{>0}} x_n y_n\Big| \le \|x\|_p \|y\|_q$$

by Hölder's inequality. So $\|\Phi_y\| \le \|y\|_q$

(cb) To show: $\|\Phi_y\| \geq \|y\|_q$

To show: There exists $x \in \ell^p$ with $|\Phi_y(x)| \geq \|x\|_p \|y\|_q$.

Let
$$x = (\mathrm{sgn}(y_1)|y_1|^{q-1}, \mathrm{sgn}(y_2)|y_2|^{q-1}, \ldots).$$

Then
$$\|x\|_p = \Big(\sum_{n\in\mathbb{Z}_{>0}} |x_n|^p\Big)^{1/p} = \Big(\sum_{n\in\mathbb{Z}_{>0}} |\mathrm{sgn}(y_n)|y_n|^{q-1}|^p\Big)^{1/p}$$

$$= \Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq-p}\Big)^{1/p} = \Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq(1-\frac{1}{q})}\Big)^{1/p}$$

$$= \Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^{pq\frac{1}{p}}\Big)^{1/p} = \Big(\Big(\sum_{n\in\mathbb{Z}_{>0}} |y_n|^q\Big)^{\frac{1}{q}}\Big)^{q\cdot\frac{1}{p}}$$

$$= \|y\|_q^{q\cdot\frac{1}{p}} = \|y\|_q^{q(1-\frac{1}{q})} = \|y\|_q^{q-1}.$$

So
$$|\Phi_y(x)| = \Big|\sum_{n\in\mathbb{Z}_{>0}} x_n y_n\Big| = \Big|\sum_{n\in\mathbb{Z}_{>0}} (\mathrm{sgn}(y_n)|y_n|)(\mathrm{sgn}(y_n)|y_n|^{q-1})\Big|$$

$$= \sum_{n\in\mathbb{Z}_{>0}} |y_n|^q = \|y\|_q^q = \|y\|_q \|y\|_q^{q-1} = \|y\|_q \cdot \|x\|_p.$$

So $\|\Phi_y\| \geq \|y\|_q$.

So $\|\Phi_y\| = \|y\|_q$.