Lecture 31: Metric and Hilbert Spaces — Standard matrix operators

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Let $H = \mathbb{C}^n$ with the standard inner product.

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}.$$

Let $B \in M_n(\mathbb{C})$,

$$A = B^* B, \quad \text{where } B^* = \overline{B}^t.$$

Then
$$A^* = (B^* B)^* = B^* (B^*)^* = B^* B = A,$$
so that A is self adjoint.

By a theorem from 1st year (why is this true?):

there exists $K \in M_n(\mathbb{C})$ with $K\overline{K}^t = 1$ and

$$KAK^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

<u>Proposition</u> (a) $\|A\| = \|KAK^{-1}\|$.

(b) If γ is the largest eigenvalue of A then
$$\|B\| = \sqrt{\gamma}.$$

<u>Proof</u> (1) If $x \in V$ then, since $K^* K = 1$ (K is unitary)

then $\|Kx\|^2 = \langle Kx, Kx \rangle = \langle x, K^* K x \rangle = \langle x, x \rangle = \|x\|^2.$

So $\|Kx\| = \|x\|$.

(2) If $x \in V$ then

$$\|KAK^{-1}x\| = \|AK^{-1}x\| \le \|A\|\,\|K^{-1}x\| = \|A\|\cdot\|x\|.$$

So $\|KAK^{-1}\| \le \|A\|$.

Since $\|K^{-1}(KAK^{-1})K\| \le \|KAK^{-1}\|$ then

$$\|A\| \le \|KAK^{-1}\|.$$

So $\|KAK^{-1}\| = \|A\|$.

(3) If $x \in V$ and $Ax = \lambda x$ then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^*Bx\rangle = \langle x, \lambda x\rangle = \lambda \|x\|^2.$$

Since $\|Bx\|^2 \in \mathbb{R}_{\ge 0}$ and $\|x\|^2 \in \mathbb{R}_{\ge 0}$ then $\lambda \in \mathbb{R}_{\ge 0}$

and $\|Bx\|^2 = \sqrt{\lambda}\,\|x\|$. So $\|B\| \ge \sqrt{\lambda}$.

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(4) If $\{a_1, a_2, \ldots, a_n\}$ is a basis of $V = \mathbb{C}^n$ consisting of eigenvectors of A and

$$x = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

is a vector in V then

(2) If $x \in V$ then

$$\|KAK^{-1}x\| = \|AK^{-1}x\| \le \|A\|\,\|K^{-1}x\| = \|A\|\,\|x\|.$$

So $\|KAK^{-1}\| \le \|A\|$.

Since $\|K^{-1}(KAK^{-1})K\| \le \|KAK^{-1}\|$ then

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So $\|KAK^{-1}\| = \|A\|$.

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Since $\|Bx\|^2 \in \mathbb{R}_{\ge 0}$ and $\|x\|^2 \in \mathbb{R}_{\ge 0}$ then $\lambda \in \mathbb{R}_{\ge 0}$

and $\|Bx\|^{2} = \sqrt{\lambda}\,\|x\|$. So $\|B\| \ge \sqrt{\lambda}$.

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(4) If $\{a_1, a_2, \ldots, a_n\}$ is a basis of $V = \mathbb{C}^n$ consisting of eigenvectors of A and

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is a vector in V then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^*Bx\rangle$$

$$= \langle x, A(x_1 a_1 + \cdots + x_n a_n)\rangle$$

$$= \langle x_1 a_1 + \cdots + x_n a_n, x_1 \lambda_1 a_1 + \cdots + x_n \lambda_n a_n\rangle$$

$$= \lambda_1 |x_1|^2 + \cdots + \lambda_n |x_n|^2$$

$$\leq \max\{\lambda_1, \ldots, \lambda_n\} (|x_1|^2 + \cdots + |x_n|^2)$$

$$= \max\{\lambda_1, \ldots, \lambda_n\} \langle x, x\rangle$$

$$= \max\{\lambda_1, \ldots, \lambda_n\} \|x\|^2 = \gamma \|x\|^2.$$

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<u>Theorem</u> Let $T: H \to H$ be a compact linear operator. Let $\lambda_1, \lambda_2, \ldots$ be distinct eigenvalues of T. Then

$$\lim_{k \to \infty} \lambda_k = 0.$$

<u>Proof</u>: Proof by contrapositive.

Assume $\lim_{k \to \infty} \lambda_k \neq 0$.

Then there exists a subsequence $\lambda_{n_1}, \lambda_{n_2}, \ldots$ and $c \in \mathbb{R}_{>0}$ with $|\lambda_{n_j}| > c$ for $j \in \mathbb{Z}_{>0}$.

To show: T is not compact.

To show: There exists a sequence $e_1, e_2, \ldots$ in H with $\|e_i\| = 1$ such that $(Te_1, Te_2, \ldots)$ does not have a cluster point.

Let $e_1, e_2, \ldots$ be a sequence in H with

$\|e_i\| = 1$ and $Te_j = \lambda_{n_j} e_j$.

Since $\lambda_i \neq \lambda_j$ then $\langle e_i, e_j \rangle = 0$ when $i \neq j$.

Then
$$\|Te_k - Te_\ell\|^2 = \|\lambda_{n_k} e_k - \lambda_{n_j} e_\ell\|^2$$
$$= |\lambda_{n_k}|^2 \|e_k\|^2 + |\lambda_{n_\ell}| \|e_\ell\|^2$$

$$= |\lambda_{n_k}|^2 + |\lambda_{n_\ell}|^2 > 2c^2.$$

So $Te_{n_1}, Te_{n_2}, \ldots$ has no Cauchy subsequence.

So $Te_{n_1}, Te_{n_2}, \ldots$ has no convergent subsequence.

So T is not a compact operator.