Lecture 30: Metric and Hilbert spaces

Let H be a Hilbert space and $T: H \to H$ a bounded linear operator.

$$X_\lambda = \{v \in H \mid Tv = \lambda v\} = \ker(T - \lambda)$$

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid X_\lambda \neq 0\}$$

Modulo the assignment we showed:

Theorem: If $T: H \to H$ is bounded self adjoint compact then

$$H = \overline{\left(\bigoplus_{\lambda \in \sigma_p(T)} X_\lambda\right)}$$

Ass 2 Q2: If $T: H \to H$ is self adjoint and $X_\lambda \neq 0$ then $\lambda \in \mathbb{R}$ (really $\lambda = \bar{\lambda}$).

Ass 2 Q2: If $T: H \to H$ is self adjoint and $\lambda \neq \gamma$ then $X_\lambda \perp X_\gamma$.

Ass 2 Q3: If $T: H \to H$ is compact and $\lambda \neq 0$ then $\dim(X_\lambda)$ is finite.

So

$$\sigma_p(T) \subseteq \mathbb{R}.$$

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More is true:

If $T: H \to H$ is self adjoint and $u \in H$ then

$$\langle Tu, u\rangle = \langle u, Tu\rangle = \overline{\langle Tu, u\rangle} \quad \text{so that}$$

$\langle Tu, u\rangle \in \mathbb{R}$. Let

$$m = \inf\{\langle Tu, u\rangle \mid u \in H \text{ and } \|u\| = 1\}$$

$$M = \sup\{\langle Tu, u\rangle \mid u \in H \text{ and } \|u\| = 1\}$$

Then, see Bressan Lemma 6.3,

$$\sigma_p(T) \subseteq [m, M]$$

and $\displaystyle \|T\| = \max\{|m|, |M|\} = \sup\left\{ |\langle Tu, u\rangle| \;\middle|\; \begin{matrix} u \in H \\ \|u\| = 1 \end{matrix} \right\}$

<u>$\langle Tu, u\rangle$ and Cauchy-Schwarz</u>

$$|\langle Tu, u\rangle| \leq \|Tu\| \cdot \|u\| \text{ and } \theta = \cos^{-1}\left(\frac{\langle Tu, u\rangle}{\|Tu\| \cdot \|u\|}\right)$$

is the "angle between u and Tu"

If $\theta = 0$ or $\theta = \pi$ then u is an eigenvector!

If $|\langle Tu, u\rangle|$ is maximal then

perhaps $\displaystyle \frac{|\langle Tu, u\rangle|}{\|Tu\| \cdot \|u\|} = \frac{\|T\|}{\|T\| \cdot 1} = 1$

and, if it is, then $\theta = 0$ or $\theta = \pi$.