Metric and Hilbert Spaces Lecture 2 27.07.2016 (1)
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(S) Functions are for comparing sets.

(T) Continuous functions are for comparing topological spaces

(U) Uniformly continuous functions are for comparing uniform spaces.

Let X and Y be sets.

A function $f : X \to Y$ is a subset $\Gamma \subseteq X \times Y$ such that

(1) If $x \in X$ then there exists $y \in Y$ such that $(x, y) \in \Gamma$

(2) If $x \in X$ and $y_1, y_2 \in Y$ and $(x, y_1) \in \Gamma$ and $(x, y_2) \in \Gamma$ then $y_1 = y_2$.

Write
$$\Gamma = \{(x, f(x)) \mid x \in X\}.$$

Let $f : X \to Y$ and $g : Y \to Z$ be functions
The composition $g \circ f$ is the function $g \circ f : X \to Z$ given by $(g \circ f)(x) = g(f(x))$.

Let X be a set. The <u>identity function</u> is

$$id_X : X \to X \quad \text{given by} \quad id_X(x) = x.$$

Let $f: X \to Y$ be a function.

An <u>inverse function to f</u> is a function $g: Y \to X$ such that

$$g \circ f = id_X \quad \text{and} \quad f \circ g = id_Y.$$

The function $f: X \to Y$ is <u>injective</u> if f satisfies:

if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$ then $x_1 = x_2$.

The function $f: X \to Y$ is <u>surjective</u> if f satisfies:

if $y \in Y$ then there exists $x \in X$ such that $f(x) = y$.

The function $f: X \to Y$ is <u>bijective</u> if it is injective and surjective.

<u>Theorem</u> Let $f: X \to Y$ be a function.

An inverse function to f exists

if and only if

f is bijective.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. A <u>continuous function from X to Y</u> is a function $f: X \to Y$ such that

if $V \in \mathcal{U}$ then $f^{-1}(V) \in \mathcal{T}$.

Recall that

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

and that $f(V)$ has nothing to do with an inverse function to f.

Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniform spaces. A <u>uniformly continuous function from X to Y</u> is a function $f: X \to Y$ such that

if $F \in \mathcal{F}$ then $(f \times f)^{-1}(F) \in \mathcal{E}$.

Here,

$$(f \times f)^{-1}(F) = \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in F\}.$$

<u>In English</u>:

continuous: Inverse images of open sets are open.

uniformly continuous: Inverse images of entourages are entourages.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces.

An <u>isomorphism</u>, or <u>homeomorphism</u>, is a continuous function $f: X \to Y$ such that

(a) The inverse function $f^{-1}: Y \to X$ exists

(b) The inverse function to f is continuous.

Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniform spaces.

An <u>isomorphism</u> <u>of uniform spaces</u> is a uniformly continuous function $f: X \to Y$ such that

(a) The inverse function $f^{-1}: Y \to X$ exists

(b) The inverse function $f^{-1}: Y \to X$ is uniformly continuous.

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Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniform spaces and let

$f: X \to Y$ be a uniformly continuous function.

Let $\mathcal{T}$ be the uniform space topology on X, and $\mathcal{U}$ the uniform space topology on Y.

Then

$f: X \to Y$ is a continuous function.

<u>Proof</u>: To show: If $V \in \mathcal{U}$ then $f^{-1}(V) \in \mathcal{T}$.

Assume $V \in \mathcal{U}$.

To show: $f^{-1}(V) \in \mathcal{T}$.

To show: If $x \in f^{-1}(V)$ then x is an interior point of $f^{-1}(V)$.

Assume $x \in f^{-1}(V)$.

Then $f(x) \in V$.

Since V is open $f(x)$ is an interior point of V.

So there exists $F \in \mathcal{F}$ with $V \ni B_F(f(x))$.

Since f is uniformly continuous $(f\times f)^{-1}(F) \in \mathcal{E}$.

Let $E = (f\times f^{-1})(F)$.

To show $f^{-1}(V) \ni B_E(x)$.

Assume $x_1 \in B_E(x)$.

Then $(x, x_1) \in E$.

So $(f(x), f(x_1)) \in F$.

So $f(x_1) \in B_F(f(x))$.

So $f(x_1) \in V$.

So $x_1 \in f^{-1}(V)$

So $B_E(x) \subseteq f^{-1}(V)$.

So $f^{-1}(V)$ is open.. ~~and x is an~~