Lecture 29: Metric and Hilbert Spaces

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Eigenvalues and eigenvectors

Let H be a Hilbert space and $T: H \to H$ a linear operator.

The λ-eigenspace of T is

$$X_\lambda = \{v \in H \mid Tv = \lambda v\} = \ker(T - \lambda).$$

The point spectrum of T is

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid X_\lambda \neq 0\}.$$

Let

$$W = \bigoplus_{\lambda \in \sigma_p(T)} X_\lambda .$$

W is the span of the eigenvectors of T.

- W is a T-submodule of H:

  If $w \in W$ then $Tw \in W$.

- If $T: H \to H$ is self adjoint then $W^\perp$ is a T-submodule of H.

Proof: To show: If $x \in W^\perp$ then $Tx \in W^\perp$.

Assume $x \in W^\perp$

To show: $Tx \in W^\perp$

To show: If $w \in W$ then $\langle Tx, w \rangle = 0$.

Assume $w \in W$.

To show: $\langle Tx, w\rangle = 0$.

$$\begin{aligned}\langle Tx, w\rangle &= \langle x, Tw\rangle, \text{ since } T \text{ is self adjoint,} \\ &= 0, \text{ since } Tw \in W \text{ and } x \in W^\perp. \ //\end{aligned}$$

- $W^\perp \supseteq \overline{W}^\perp$

<u>Proof</u> $$\begin{aligned}\overline{W}^\perp &= \{x \in H \mid \text{If } w \in \overline{W} \text{ then } \langle x, w\rangle = 0\} \\ &\subseteq \{x \in H \mid \text{If } w \in W \text{ then } \langle x, w\rangle = 0\} \\ &= W^\perp\end{aligned}$$

since $W \subseteq \overline{W}$.

- If $T: H \to H$ is a compact operator (and $T: H \to H$ is self adjoint so that $W^\perp$ is a T-submodule) then

$T: W^\perp \to W^\perp$ is a compact operator.

<u>Proof</u>: To show: If $(u_1, u_2, \ldots)$ is a sequence in $W^\perp$ with $\|u_i\| = 1$ then $(Tu_1, Tu_2, \ldots)$ has a cluster point in $W^\perp$.

Assume $(u_1, u_2, \ldots)$ is a sequence in $W^\perp$ with $\|u_i\| = 1$.

Then $(u_1, u_2, \ldots)$ is a sequence in H with $\|u_i\| = 1$.

Since T is compact then

$(Tu_1, Tu_2, \ldots)$ has a cluster point z in H.

Since $(Tu_1, Tu_2, \ldots)$ is a sequence in $W^\perp$

and $W^\perp$ is closed then $z \in W^\perp$.

So $(Tu_1, Tu_2, \ldots)$ has a cluster point in $W^\perp$.

So $T : W^\perp \to W^\perp$ is a compact operator. //.

- If $v \in W^\perp$ is an eigenvector of $T : W^\perp \to W^\perp$ then $v = 0$.

<u>Proof</u>: Assume $v \in W^\perp$ and $Tv = \lambda v$ with $\lambda \in \mathbb{C}$.

Then $v \in X_\lambda$ and so $v \in W$.

So $\langle v, v \rangle = 0$. So $v = 0$. //.

<u>Theorem</u> If $H \neq 0$ and $T : H \to H$ is a bounded self adjoint compact linear operator then T has an (nonzero) eigenvector.

<u>Corollary</u>: If $T : H \to H$ is a bounded self adjoint linear operator then

$$W = \bigoplus_{\lambda \in \sigma_p(T)} X_\lambda \text{ is dense in } H.$$

Proof By the orthogonal decomposition theorem $H = \overline{W} \oplus \overline{W}^{\perp}$.

If $W^{\perp} \neq 0$ then $T: W^{\perp} \to W^{\perp}$ has an nonzero eigenvector which is a contradiction to (*).

So $W^{\perp} = 0$.

So $\overline{W}^{\perp} = 0$ (since $\overline{W}^{\perp} \subseteq W^{\perp}$).

So $H = \overline{W}$. //

Remark: By assignment 2 Q 2,

$\sigma_p(T) \subseteq \mathbb{R}$ when $T: H \to H$ is self adjoint.

More is true:

Recall, if $u \in H$ and $T: H \to H$ is self adjoint then $\langle Tu, u\rangle = \langle u, Tu\rangle = \overline{\langle Tu, u\rangle}$ so that $\langle Tu, u\rangle \in \mathbb{R}$.

Let

$$m = \inf\{\langle Tu, u\rangle \mid u \in H \text{ and } \|u\| = 1\}$$
$$M = \sup\{\langle Tu, u\rangle \mid u \in H \text{ and } \|u\| = 1\}.$$

Then, see Bressan Lemma 6.3,

$$\sigma_p(T) \subseteq [m, M].$$

and $\|T\| = \max\{|m|, |M|\} = \sup\{|\langle Tu, u\rangle| \mid u \in H \text{ and } \|u\| = 1\}$