Univ. Melbourne
A. Ram.

Lecture 27: Metric and Hilbert spaces
Orthogonal decomposition theorem

Let H be a Hilbert space. Let W be a subspace.

$H = W \oplus W^\perp$ if and only if W is closed.

The real point is to construct

$$P_W : H \to W$$
$$x \mapsto P_W(x)$$

the projection onto W.

Construction 1: $P_W(x) = y$ where

$y \in W$ and $d(x,y) = \inf\{d(x,w) \mid w \in W\}$.

Construction 2: $P_W(x) = y$ where

$y \in W$ and $x - y \in W^\perp$.

Claim

Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H

Let $W = \mathbb{K}\text{span}\{a_1, a_2, \ldots\}$ and $\overline{W} = \overline{\mathbb{K}\text{span}\{a_1, a_2, \ldots\}}$

Then

$$P_{\overline{W}}(x) = \sum_{n=1}^{\infty} \langle x, a_n\rangle a_n$$

Step 1 (Bessel's Inequality)

If $x \in H$ then $\sum_{i=1}^{\infty} |\langle x, a_n\rangle|^2 \le \|x\|^2$.

Assume $x \in H$.

To show: $\sum_{i=1}^{\infty} |\langle x, a_n\rangle|^2 \le \|x\|^2$.

To show: $\lim_{k\to\infty} \left(\sum_{i=1}^{k} |\langle x, a_n\rangle|^2\right) \le \|x\|^2$.

To show: If $k \in \mathbb{Z}_{>0}$ then $\sum_{i=1}^{k} |\langle x, a_n\rangle|^2 \le \|x\|^2$.

Assume $k \in \mathbb{Z}_{>0}$ and let

$$x_k = \sum_{i=1}^{k} \langle x, a_n\rangle a_n.$$

Then

$$\|x_k\|^2 = \left\langle \sum_{i=1}^{k} \langle x, a_n\rangle a_n, \sum_{m=1}^{k} \langle x, a_m\rangle a_m \right\rangle$$

$$= \sum_{m,n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_m\rangle} \langle a_n, a_m\rangle$$

$$= \sum_{n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_n\rangle} = \sum_{n=1}^{k} |\langle x, a_n\rangle|^2.$$

To show: $\|x_k\|^2 \le \|x\|^2$

Then

$$\langle x - x_k, x_k \rangle = \langle x, x_k \rangle - \langle x_k, x_k \rangle$$

$$= \sum_{n=1}^{k} \langle x, a_n \rangle \overline{\langle x, a_n \rangle} - \sum_{n=1}^{k} \langle x, a_n \rangle \overline{\langle x, a_n \rangle}$$

$$= 0.$$

and

$$\|x\|^2 = \langle x, x \rangle = \langle x_k + (x - x_k), x_k + (x - x_k) \rangle$$

$$= \langle x_k, x_k \rangle + 0 + 0 + \langle x - x_k, x - x_k \rangle$$

$$= \|x_k\|^2 + \|x - x_k\|^2$$

So $\|x_k\|^2 \le \|x\|^2$.

So $\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^2 = \lim_{k \to \infty} \|x_k\|^2 \le \|x\|^2$.

<u>Step 2</u> To show: $\sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$ exists in H.

To show: $\lim_{k \to \infty} x_k$ exists in H.

Since H is complete we need

To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence in H.

By Bessel's inequality,

$(\|x_1\|, \|x_2\|, \ldots)$ is an increasing sequence in $\mathbb{R}_{\ge 0}$ which is bounded by $\|x\|$.

So $(\|x_1\|, \|x_2\|, \ldots)$ converges. Let $y = \lim_{k\to\infty} \|x_k\|$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$.

Let $N \in \mathbb{Z}_{>0}$ be such that if $k \in \mathbb{Z}_{\geq N}$ then $|y^2 - \|x_k\|^2| < \frac{\varepsilon}{2}$

Assume $r, s \in \mathbb{Z}_{\geq N}$.

To show: $\|x_r - x_s\|^2 < \varepsilon$.

$$\|x_r - x_s\|^2 = \Big\|\sum_{j=1}^{r} \langle x, a_j\rangle a_j - \sum_{j=1}^{s} \langle x, a_j\rangle a_j\Big\|^2$$

$$= \Big\|\sum_{j=r+1}^{s} \langle x, a_j\rangle a_j\Big\|^2 = \sum_{j=r+1}^{s} |\langle x, a_j\rangle|^2$$

$$= \big|\|x_s\|^2 - \|x_r\|^2\big| = \big|\|x_s\|^2 - y^2 + y^2 - \|x_r\|^2\big|$$

$$\leq \big|\|x_s\|^2 - y^2\big| + \big|y^2 - \|x_r\|^2\big|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $(x_1, x_2, \ldots)$ is a Cauchy sequence in H.
So $\lim_{k\to\infty} x_k$ exists in H.

So $\sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$ exists in H.

Let $P_{\overline{W}}(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$.

Since $x_k \in W$ and $P_{\overline{W}}(x) = \lim_{k \to \infty} x_k$ then

$P_{\overline{W}}(x) \in \overline{W}$.

Step 3 To show: $x - P_{\overline{W}}(x) \in \overline{W}^{\perp}$.

To show: If $b \in \overline{W}$ then $\langle x - P_{\overline{W}}(x), b \rangle = 0$.

Assume $b \in \overline{W}$.

Let $(b_1, b_2, \ldots)$ be a sequence in W with $b = \lim_{k \to \infty} b_k$.

To show: $\langle x - P_{\overline{W}}(x), b \rangle = 0$.

Using that $\langle x - P_{\overline{W}}(x), \cdot \rangle : H \to \mathbb{K}$ is continuous,

$$\langle x - P_{\overline{W}}(x), b \rangle = \langle x - P_{\overline{W}}(x), \lim_{n \to \infty} b_n \rangle$$

$$= \lim_{n \to \infty} \langle x - P_{\overline{W}}(x), b_n \rangle.$$

Since $b_n \in W$ there exist $c_1, c_2, \ldots, c_\ell \in \mathbb{C}$ with

$b_n = c_1 a_1 + \cdots + c_\ell a_\ell$.

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Using that $\langle \cdot, a_r \rangle : H \to \mathbb{C}$ is continuous, and that $\langle x_k, a_r \rangle = \langle x, a_r \rangle$ for $k \geq r$, then

$$\begin{aligned}
\langle x - P(x), a_r \rangle &= \langle x, a_r \rangle - \langle P(x), a_r \rangle \\
&= \langle x, a_r \rangle - \langle \lim_{k\to\infty} x_k, a_r \rangle \\
&= \langle x, a_r \rangle - \lim_{k\to\infty} \langle x_k, a_r \rangle \\
&= \langle x, a_r \rangle - \langle x, a_r \rangle = 0.
\end{aligned}$$

So

$$\begin{aligned}
\langle x - P_W(x), b_n \rangle &= \langle x - P_W(x), c_1 a_1 + \cdots + c_l a_l \rangle \\
&= \bar{c}_1 \langle x - P_W(x), a_1 \rangle + \cdots + \bar{c}_l \langle x - P_W(x), a_l \rangle \\
&= 0.
\end{aligned}$$

So

$$\langle x - P_{\overline{W}}(x), b \rangle = \lim_{n\to\infty} \langle x - P_{\overline{W}}(x), b_n \rangle = \lim_{n\to\infty} 0 = 0$$

So $x - P_{\overline{W}}(x) \in \overline{W}^{\perp}$.