Univ. Melbourne
A. Ram.

Lecture 26: Metric and Hilbert Spaces

Theorem (Riesz representation Theorem)

Let $(H, \langle\,,\,\rangle)$ be a Hilbert space. Then

$$\begin{aligned} H &\to H^* \\ x &\mapsto \varphi_x : H \to \mathbb{K} \\ &\qquad y \mapsto \langle x, y\rangle \end{aligned}$$

is a bijective isometry.

Proof To show: (a) Ψ is a linear transformation.

(b) Ψ is bounded

(c) Ψ is an isometry.

(d) Ψ is injective

(e) Ψ is surjective.

(a) To show: (aa) If $a, b \in H$ then $\varphi_{a+b} = \varphi_a + \varphi_b$

(ab) If $a \in H$ and $c \in \mathbb{K}$ then $\varphi_{ca} = c\varphi_a$

(aa) Assume $a, b \in H$

To show: $\varphi_{a+b} = \varphi_a + \varphi_b$.

To show: If $y \in H$ then $\varphi_{a+b}(y) = (\varphi_a + \varphi_b)(y)$.

Assume $y \in H$.

To show: $\varphi_{a+b}(y) = (\varphi_a + \varphi_b)(y)$.

$$\varphi_{a+b}(y) = \langle a+b, y\rangle = \langle a, y\rangle + \langle b, y\rangle$$

$$(\varphi_a + \varphi_b)(y) = \varphi_a(y) + \varphi_b(y) = \langle a, y\rangle + \langle b, y\rangle.$$

(ab) Assume $a \in H$ and $c \in \mathbb{K}$.

To show: $\varphi_{ca} = c\varphi_a$.

To show: If $y \in H$ then $\varphi_{ca}(y) = c\varphi_a(y)$.

Assume $y \in H$.

To show: $\varphi_{ca}(y) = c\varphi_a(y)$

$\varphi_{ca}(y) = \langle ca, y\rangle = c\langle a, y\rangle$, and

$c\varphi_a(y) = c\langle a, y\rangle$.

(c) To show: Φ is an isometry.

To show: If $x \in H$ then $\|\varphi_x\| = \|x\|$.

Assume $x \in H$.

To show: $\|\varphi_x\| = \|x\|$.

To show: (ca) $\|\varphi_x\| \le \|x\|$

(cb) $\|\varphi_x\| \ge \|x\|$.

(ca) To show: $\|\varphi_x\| \le \|x\|$.

To show: $\sup\left\{ \frac{\|\varphi_x(y)\|}{\|y\|} \,\middle|\, \begin{matrix} y \in H \\ y \ne 0 \end{matrix} \right\} \le \|x\|$

To show: If $y \in H$ and $y \ne 0$ then $\|\varphi_x(y)\| \le \|x\|\|y\|$.

Assume $y \in H$ and $y \ne 0$.

Using Cauchy-Schwartz,

$\|\varphi_x(y)\| = |\langle x, y\rangle| \le \|x\| \cdot \|y\|$.

(cb) To show: $\|\Psi_x\| \geq \|x\|$.

To show: There exists $y \in H$, $y \neq 0$ with $\|\Psi_x(y)\| = \|x\|\|y\|$.

Let $y = x$.

To show: $\|\Psi_x(y)\| = \|x\| \cdot \|y\|$.

$\|\Psi_x(x)\| = |\langle x, x\rangle| = \|x\|^2 = \|x\| \cdot \|x\|$.

So $\|\Psi_x\| \geq \|x\|$.

So $\|\Psi_x\| = \|x\|$.

So Ψ is an isometry.

(b) To show: Ψ is ~~an isometric~~ bounded.

To show: $\|\Psi\| < \infty$.

To show: $\sup\left\{ \frac{\|\Psi_x\|}{\|x\|} \;\middle|\; \begin{matrix} x \in H \\ x \neq 0 \end{matrix} \right\} < \infty$.

~~To show:~~ Since $\|\Psi_x\| = \|x\|$ then

$$\|\Psi\| = \sup\left\{ \frac{\|\Psi_x\|}{\|x\|} \;\middle|\; \begin{matrix} x \in H \\ x \neq 0 \end{matrix} \right\} = \sup\{1\} = 1.$$

So $\|\Psi\| < \infty$.

So ~~$\|\Psi\|$ is~~ Ψ is bounded.

(c) To show: Ψ is injective

To show: If $a, b \in H$ and $\Psi_a = \Psi_b$ then $a = b$.

Assume $a, b \in H$ and $\Psi_a = \Psi_b$.

To show: $a=b$.

To show: $\|a-b\|=0$.

$$\|a-b\| = \|\varphi_{a-b}\| = \|\varphi_a - \varphi_b\| = \|0\| = 0.$$

So Φ is injective.

(b) To show: Φ is surjective.

To show: If $\varphi \in H^*$ then there exists $a \in H$ with $\varphi_a = \varphi$.

Assume $\varphi \in H^*$

To show: There exists $a \in H$ with $\varphi_a = \varphi$.

Case 1: $\varphi = 0$.

Then let $a=0$ so that $\varphi_0 = 0 = \varphi$.

Case 2: $\varphi \neq 0$. $\varphi : H \to \mathbb{K}$.

Since φ is bounded then φ is continuous.

Since $\{0\}$ is closed in $\mathbb{K}$ then

$\ker\varphi = \varphi^{-1}(\{0\})$ is closed in H.

Use the orthogonal decomposition theorem

$$H = \ker\varphi \oplus (\ker\varphi)^\perp.$$

Let

$b \in (\ker\varphi)^\perp$ with $b \neq 0$ and $a = \dfrac{\overline{\varphi(b)}}{\|b\|^2} b$.

To show: $\psi_a = \varphi$.

To show: If $h \in H$ then $\psi_a(h) = \varphi(h)$.

Assume $h \in H$.

$$h = \left(h - \frac{\varphi(h)}{\varphi(a)} a\right) + \frac{\varphi(h)}{\varphi(a)} a$$

where $\frac{\varphi(h)}{\varphi(a)} a \in (\ker \varphi)^{\perp}$ and $h - \frac{\varphi(h)}{\varphi(a)} a \in \ker \varphi$

since

$$\varphi\left(h - \frac{\varphi(h)}{\varphi(a)} a\right) = \varphi(h) - \frac{\varphi(h)}{\varphi(a)} \varphi(a) = 0.$$

To show: $\psi_a(h) = \varphi(h)$

$$\psi_a(h) = \langle a, h \rangle = \left\langle a, \left(h - \frac{\varphi(h)}{\varphi(a)} a\right) + \frac{\varphi(h)}{\varphi(a)} a \right\rangle$$

$$= 0 + \overline{\frac{\varphi(h)}{\varphi(a)}} \langle a, a \rangle$$

$$= \frac{\overline{\varphi(h)}}{\overline{\frac{\varphi(b)}{\|b\|^2}} \, \overline{\varphi(b)}} \; \frac{\overline{\varphi(b)}\, \varphi(b)}{\|b\|^2 \|b\|^2} \langle b, b \rangle = \overline{\varphi(h)}.$$