Lecture 25: Metric and Hilbert spaces

Theorem Let $(H, \langle\,,\,\rangle)$ be a Hilbert space. Let W be a subspace of H. Then

W is closed if and only if $H = W \oplus W^\perp$.

Cauchy-Schwarz: If $x, y \in H$ then

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|.$$

Pythagorean theorem: If $x, y \in H$ and $\langle x, y\rangle = 0$ then $\|x\|^2 + \|y\|^2 = \|x+y\|^2$.

Parallelogram law: If $x, y \in H$ then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Definition of $W^\perp$: Let W be a <u>subset</u> of H.

$$W^\perp = \{x \in H \mid \text{if } w \in W \text{ then } \langle x, w\rangle = 0\}$$

$$= \left\{x \in H \;\middle|\; \begin{array}{l} \varphi_x : W \to \mathbb{K} \\ \quad w \mapsto \langle x, w\rangle \end{array} \text{ is the zero map}\right\}$$

$$= \{x \in H \mid \varphi_x = 0\} = \Phi^{-1}(0) = \ker \Phi$$

where
$$\Phi : H \to W^*$$
$$x \mapsto \Phi_x.$$

Projections of H onto W

$$P_W : H \longrightarrow W$$
$$x \longmapsto P_W(x).$$

Define $P_W(x) = y$ where
$y \in W$ and $d(x,y) = \inf\{ d(x,w) \mid w \in W\}$.

Define $P_W(x) = w$ where
$y \in W$ and $x - y \in W^\perp$.

Definition 1 is well defined: Let $x \in H$.

To show: (a) There exists $y \in W$ such that
$d(x,y) = \inf\{ \|x-w\| \mid w \in W\}$.

(b) If $y_1, y_2 \in W$ and $d(x,y_1) = d(x,W)$ and
$d(x,y_2) = d(x,W)$ then $y_1 = y_2$.

Definition 2 is well defined: Let $x \in H$.

To show: (a) There exists $y \in W$ such that
$x - y \in W^\perp$.

(b) If $y_1, y_2 \in W$ and $x - y_1 \in W^\perp$ and $x - y_2 \in W^\perp$
then $y_1 = y_2$.

For part (a), we will need to assume
W is closed.

Definition of $H = W \oplus V$

(a) If $x \in H$ then $x = y + v$ with $y \in W$ and $v \in V$.

(b) If $x \in H$ ~~to~~ and $x = y_1 + v_1$ and $x = y_2 + v_2$ with $y_1, y_2 \in W$ and $v_1, v_2 \in V$ then $y_1 = y_2$ and $w_1 = w_2$.

Additional facts about projections

(1) $P_W : H \to W$, $x \mapsto P_W(x)$ is a bounded linear operator.

(2) $P_{W^\perp} = id_H - P_W$

(3) If $W \neq 0$ and $W^\perp \neq 0$ then $\|P_W\| = 1$ and $\|P_{W^\perp}\| = 1$.

Additional facts about $W^\perp$

(a) $\Psi : H \to W^*$, $x \mapsto \Psi_x$ is a bounded linear operator.

Here $\Psi_x : W \to \mathbb{K}$ is given by $\Psi_x(w) = \langle x, w \rangle$.

(b) If $H = W \oplus W^\perp$ then

$$(W^\perp)^\perp = W.$$

(c) In general, $(W^\perp)^\perp \supseteq W$. Give an example where $(W^\perp)^\perp \supsetneq W$.