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Lecture 24: Metric and Hilbert spaces

Let $(H, \langle\,\rangle)$ be a Hilbert space.

Let $W \subseteq H$ be a subspace of H.

The orthogonal to W is

$$W^\perp = \{v \in V \mid \text{~~if~~ } w \in W \text{ then } \langle v, w\rangle = 0\}$$

Big theorem for next week: Let H be a Hilbert space and $W \subseteq H$ a subspace. Then W is closed if and only if

$$H = W \oplus W^\perp.$$

Note: $H = W \oplus W^\perp$ means

(a) If $v \in H$ then there exist $w \in W$ and $x \in W^\perp$ with $v = w + x$, and

(b) If $x \in W$ and $x \in W^\perp$ then $x = 0$.

An orthonormal sequence in H is a sequence $(a_1, a_2, \ldots)$ in H such that

$$\text{if } i,j \in \mathbb{Z}_{>0} \text{ then } \langle a_i, a_j\rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

"Fourier expansion"

Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H. Let

$W = \mathrm{span}\{a_1, a_2, \ldots\}$. So $\overline{W} = \overline{\mathbb{K}\text{-span}\{a_1, a_2, \ldots\}}$

Then
$$H = \overline{W} \oplus \overline{W}^\perp.$$

If $v \in H$ then there exists a unique $x \in \overline{W}^\perp$ and
$$v = \left(\sum_{i=1}^{\infty} \langle v, a_i \rangle a_i\right) + x \quad \in \overline{W} \oplus \overline{W}^\perp.$$

Gram-Schmidt

Let H be a Hilbertspace. Let $(v_1, v_2, \ldots)$ be a sequence of linearly independent vectors in H. Define

$$a_1 = \frac{v_1}{\|v_1\|} \quad \text{and} \quad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \cdots - \langle v_{n+1}, a_n \rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \cdots - \langle v_{n+1}, a_n \rangle a_n\|}.$$

for $n \in \mathbb{Z}_{>0}$. Then $(a_1, a_2, \ldots)$ is an orthonormal sequence in H.

Example $\ell^2 = \{x = (x_1, x_2, \ldots) \in \mathbb{C}^\infty \mid \|x\| < \infty\}$

where
$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \quad \text{for} \quad \begin{matrix} x = (x_1, x_2, \ldots) \\ y = (y_1, y_2, \ldots) \end{matrix}$$

and $\|x\| = \sqrt{\langle x, x\rangle}$.

Let $T: \ell^2 \to \ell^2$ be the linear operator given by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$

(a) Show that the adjoint of T is $T^*: \ell^2 \to \ell^2$ given by $T^*(x_1, x_2, \ldots) = (x_2, x_3, x_4, \ldots)$.

Proof (sketch) $Te_i = e_{i+1}$, and

$$\delta_{j,i+1} = \langle e_{i+1}, e_j\rangle = \langle Te_i, e_j\rangle = \langle e_i, T^* e_j\rangle = \delta_{i,j-1}.$$

Then
$$T\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} x_i e_{i+1} \quad \text{and}$$

$$T^*\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=2}^{\infty} x_i e_{i-1}.$$

The matrix of T in the "basis" $(e_1, e_2, \ldots)$ is

$$A = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \end{pmatrix} \quad \text{and} \quad A^t = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \end{pmatrix}$$

is the matrix of T^*. Then

$$T^*T = 1 \quad \text{but} \quad T^*T \neq 1.$$

Note: 1 means $id_H : H \to H$, $v \mapsto v$ (linear transformation).

(c) If $v = (v_1, v_2, \ldots)$ is an eigenvector of T with eigenvalue λ then

$$(0, v_1, v_2, \ldots) = Tv = \lambda v = (\lambda v_1, \lambda v_2, \ldots)$$

So $v = (0, 0, \ldots)$ and T has no nonzero eigenvectors.

(d) If $v = (v_1, v_2, \ldots)$ is an eigenvector of T^* with eigenvalue λ then

$$(v_2, v_3, \ldots) = T^*v = \lambda v = (\lambda v_1, \lambda v_2, \ldots)$$

So $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ is an eigenvector of eigenvalue λ.

If $X_\lambda = \{v \in \ell^2 \mid T^*v = \lambda v\}$ (the λ eigenspace of T^*)

then $X_\lambda = \mathbb{K} v_\lambda$ where $v_\lambda = (1, \lambda, \lambda^2, \ldots)$.

(e) What is $\|T\|$?

(f) Is T self adjoint?

(g) Is T compact?