14.09.2016
Univ. Melbourne

Lecture 23: Metric and Hilbert Spaces

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V and W be $\mathbb{K}$ vector spaces.

A <u>linear operator</u> from V to W is a function $T: V \to W$ such that

(a) if $v_1, v_2 \in V$ then $T(v_1 + v_2) = T(v_1) + T(v_2)$,

(b) if $v \in V$ and $c \in \mathbb{K}$ then $T(cv) = cT(v)$

Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed $\mathbb{K}$-vector spaces. The space of <u>bounded linear operators from V to W</u> is

$$B(V,W) = \{\text{linear operators } T: V \to W \mid \|T\| < \infty\}$$

where

$$\|T\| = \sup\left\{ \frac{\|Tv\|_W}{\|v\|_V} \;\middle|\; \begin{matrix} v \in V \\ v \neq 0 \end{matrix} \right\}$$

A <u>Banach space</u> is a normed vector space $(V, \|\ \|_V)$ which is complete (with metric $d(v,w) = \|w - v\|$)

A <u>Hilbert space</u> is an inner product space $(V, \langle\ ,\ \rangle)$ which is complete (with norm $\|v\| = \sqrt{\langle v, v\rangle}$)

<u>Theorem</u> Let V and W be normed vector spaces. If W is a Banach space then $B(V,W)$ is a Banach space.

<u>Theorem</u> Let V and W be normed vector spaces. and let $T: V \to W$ be a linear operator.

(a) $T \in B(V,W)$ if and only if T is continuous.

(b) T is continuous if and only if T is uniformly continuous

<u>Duals and adjoints</u>

Let $(V, \| \ \|)$ be a normed $\mathbb{K}$-vector space.

The <u>dual of V</u> is

$$V^* = B(V, \mathbb{K}) = \left\{ \varphi : V \to \mathbb{K} \;\middle|\; \begin{array}{l} \varphi \text{ is linear and} \\ \|\varphi\| < \infty \end{array} \right\}$$

Let $(V, \| \ \|_V)$ and $(W, \| \ \|_W)$ be normed $\mathbb{K}$-vector spaces and $T: V \to W$ a bounded linear operator.

The <u>adjoint of T</u> is

$T^*: W^* \to V^*$ given by $(T^*\psi)(v) = (\psi \circ T)(v)$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, dotted, "T^*\psi."'] & W \arrow[d, "\psi"] \\ & \mathbb{K}. \end{tikzcd}$$

<u>Theorem</u> Let H be a Hilbert space. Then

$$\begin{array}{ccc} H & \xrightarrow{\psi} & H^* \\ x & \longmapsto & \psi_x \end{array}$$ is an isomorphism.

<u>Proposition</u> Let H_1 and H_2 be Hilbert spaces. Let $T: H_1 \to H_2$ be a bounded linear operator. Then $T^*: H_2 \to H_1$ is given by

$$\langle Tx, y \rangle_2 = \langle x, T^*y \rangle_1, \quad \text{for } x \in H_1, \ y \in H_2$$

Let H be a Hilbert space and let

$T: H \to H$ be a bounded linear operator.

(a) T is <u>self adjoint</u> if $T = T^*$.

(b) T is <u>positive</u> if $T = T^*$ <u>and</u>

if $x \in H$ then $\langle Tx, x \rangle \in \mathbb{R}_{\geq 0}$.

(c) T is <u>unitary</u> if $TT^* = T^*T = 1$.

(d) T is an <u>isometry</u> if T satisfies

if $x, y \in H$ then $\langle Tx, Ty \rangle_2 = \langle x, y \rangle_1$

Let X be a normed vector space.

A bounded linear operator $T : X \to X$ is <u>compact</u>

if $\overline{\{Tx \mid \|x\| = 1\}}$ is compact

i.e. if $(x_1, x_2, \ldots)$ is a sequence in $\{x \in H \mid \|x\| = 1\}$
then the sequence $(Tx_1, Tx_2, \ldots)$ has a cluster point.