

## Lecture 23: Metric and Hilbert Spaces

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Let  $K$  be  $\mathbb{R}$  or  $\mathbb{C}$  and let  $V$  and  $W$  be  $K$  vector spaces.

A linear operator from  $V$  to  $W$  is a function  $T: V \rightarrow W$  such that

$$(a) \text{ if } v_1, v_2 \in V \text{ then } T(v_1 + v_2) = T(v_1) + T(v_2),$$

$$(b) \text{ if } v \in V \text{ and } c \in K \text{ then } T(cv) = cT(v)$$

Let  $(V, \|\cdot\|_V)$  and  $(W, \|\cdot\|_W)$  be normed  $K$ -vector spaces. The space of bounded linear operators from  $V$  to  $W$  is

$$B(V, W) = \{ \text{linear operators } T: V \rightarrow W \mid \|T\| < \infty \}$$

where

$$\|T\| = \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \mid v \in V, v \neq 0 \right\}$$

A Banach space is a normed vector space  $(V, \|\cdot\|_V)$  which is complete (with metric  $d(v, w) = \|w - v\|$ )

A Hilbert space is an inner product space  $(V, \langle \cdot, \cdot \rangle)$

which is complete (with norm  $\|v\| = \sqrt{\langle v, v \rangle}$ )

Theorem Let  $V$  and  $W$  be normed vector spaces. If  $W$  is a Banach space then

$B(V, W)$  is a Banach space.

Theorem Let  $V$  and  $W$  be normed vector spaces. <sup>(2)</sup>  
and let  $T: V \rightarrow W$  be a linear operator.

- (a)  $T \in B(V, W)$  if and only if  $T$  is continuous.
- (b)  $T$  is continuous if and only if  $T$  is uniformly continuous

Duals and adjoints

Let  $(V, \|\cdot\|)$  be a normed  $\mathbb{K}$ -vector space.

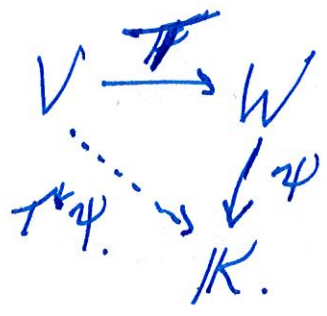
The dual of  $V$  is

$$V^* = B(V, \mathbb{K}) = \left\{ \varphi : V \rightarrow \mathbb{K} \mid \begin{array}{l} \varphi \text{ is linear and} \\ \|\varphi\| < \infty \end{array} \right\}$$

Let  $(V, \|\cdot\|_V)$  and  $(W, \|\cdot\|_W)$  be normed  $\mathbb{K}$ -vector spaces  
and  $T: V \rightarrow W$  a bounded linear operator.

The adjoint of  $T$  is

$$T^*: W^* \rightarrow V^* \text{ given by } (T^*\varphi)(v) = (\varphi \circ T)(v)$$





Theorem Let  $H$  be a Hilbert space. Then

$H \xrightarrow{\psi} H^*$  is an isomorphism.  
 $x \mapsto \psi_x$

Proposition Let  $H_1$  and  $H_2$  be Hilbert spaces  
Let  $T: H_1 \rightarrow H_2$  be a bounded linear operator.

Then  $T^*: H_2 \rightarrow H_1$  is given by

$$\langle Tx, y \rangle_2 = \langle x, T^*y \rangle_1 \quad \text{for } x \in H_1, y \in H_2$$

Let  $H$  be a Hilbert space and let

$T: H \rightarrow H$  be a bounded linear operator.

(a)  $T$  is self adjoint if  $T = T^*$ .

(b)  $T$  is positive if  $T = T^*$  and

if  $x \in H$  then  $\langle Tx, x \rangle \in \mathbb{R}_{\geq 0}$ .

(c)  $T$  is unitary if  $TT^* = T^*T = I$ .

(d)  $T$  is an isometry if  $T$  satisfies

if  $x, y \in H$  then  $\langle Tx, Ty \rangle_2 = \langle x, y \rangle_1$ .

Let  $X$  be a normed vector space.

A bounded linear operator  $T: X \rightarrow X$  is compact

if  $\overline{\{Tx \mid \|x\|=1\}}$  is compact

i.e. if  $(x_1, x_2, \dots)$  is a sequence in  $\{x \in H \mid \|x\|=1\}$

then the sequence  $(Tx_1, Tx_2, \dots)$  has a cluster point.