Lecture 20: Metric and Hilbert spaces

Let (X,d) be a metric space.

A contraction mapping is a function $f: X \to X$ such that there exists $\alpha \in (0,1)$ such that

if $x, y \in X$ then $d(f(x), f(y)) \le \alpha d(x,y)$.

A fixed point of $f: X \to X$ is an element $x \in X$ such that $f(x) = x$.

Theorem: Banach fixed point theorem

Let (X,d) be a complete metric space. and let $f: X \to X$ be a contraction mapping.

Let $x \in X$ and let $x_1, x_2, \ldots$ be the sequence

$$x_1 = f(x),\quad x_2 = f(f(x)),\quad x_3 = f(f(f(x))),\ \ldots$$

Then $x_1, x_2, \ldots$ converges and

$p = \lim_{n\to\infty} x_n$ is the unique fixed point of f.

Proof To show: (a) $p = \lim_{n\to\infty} x_n$ exists

(b) $f(p) = p$

(c) If q is a fixed point of f then $q = p$.

(a) Using that X is complete,

To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>\ell}$ then $d(x_m, x_n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let ℓ be the smallest integer in $\mathbb{Z}_{>0}$ such that

$$\frac{\alpha^{\ell} d(f(x), x)}{1-\alpha} < \varepsilon \qquad \left(\text{so } \alpha^{\ell} < \frac{\varepsilon(1-\alpha)}{d(f(x), x)} \right).$$

To show: If $m, n \in \mathbb{Z}_{>\ell}$ then $d(x_m, x_n) < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{>\ell}$. Assume $m < n$.

To show: $d(x_m, x_n) < \varepsilon$.

Since
$$d(x_2, x_1) = d(f^2(x), f(x)) \le \alpha\, d(f(x), x)$$
$$d(x_3, x_2) = d(f^3(x), f^2(x)) \le \alpha\, d(f^2(x), f(x)) \le \alpha^2 d(f(x), x)$$
$$d(x_4, x_3) = d(f^4(x), f^3(x)) \le \alpha\, d(f^3(x), f^2(x)) \le \alpha^3 d(f(x), x)$$
$\ldots$

then

$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$

$\leq \alpha^m d(f(x), x) + \alpha^{m+1} d(f(x), x) + \cdots + \alpha^{n-1} d(f(x), x)$

$= (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) d(f(x), x)$

$\leq \alpha^m (1 + \alpha + \alpha^2 + \cdots) d(f(x), x) = \dfrac{\alpha^m}{1-\alpha} d(f(x), x) < \varepsilon.$

So $(x_1, x_2, \ldots)$ is a Cauchy sequence in X.

So $(x_1, x_2, \ldots)$ converges in X

So $p = \lim_{n\to\infty} x_n$ exists in X.

(b) To show: $f(p) = p$.

To show: $d(f(p), p) = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $d(f(p), p) < \varepsilon$

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: $d(f(p), p) < \varepsilon$.

Since $\lim_{n\to\infty} x_n = p$, there exists $N \in \mathbb{Z}_{>0}$

such that if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, p) < \frac{\varepsilon}{2}$

Then
$$d(f(p),p) \le d(f(p), x_{N+1}) + d(x_{N+1}, p)$$
$$\le \alpha\, d(p, x_N) + d(x_{N+1}, p)$$
$$< \alpha \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

So $d(f(p),p) = 0$.

So $f(p) = p$.

(c) To show: If q is a fixed point of f then $q = p$.

To show: If $q \in X$ and $f(q) = q$ then $q = p$.

Assume $q \in X$ and $f(q) = q$.

To show: $q = p$.

~~To~~ show: $d(q,p) = 0$.

$$d(q,p) = d(f(q), f(p))$$
$$\le \alpha\, d(q,p)$$

So $(1-\alpha)\, d(q,p) \le 0$.

Since $(1-\alpha)\, d(q,p) \ge 0$ and $(1-\alpha)\, d(q,p) \le 0$ then
$(1-\alpha)\, d(q,p) = 0$.

Since $(1-\alpha) \ne 0$ then $d(q,p) = 0$.

So $p = q$. //