Lecture 21: Metric and Hilbert Spaces

Theorem Let $(V, \|\|)$ be a normed vector space and $d: V\times V \to \mathbb{R}_{\geq 0}$ on V given by
$$d(x,y) = \|y-x\|.$$
Then V is a complete metric space if and only if V satisfies

If $(a_1, a_2, \ldots)$ is a sequence in V and $\sum_{i\in\mathbb{Z}_{>0}} \|a_i\|$ converges then $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

Proof $\Rightarrow$ Assume V is complete.

To show: V satisfies $(*)$.

Assume $(a_1, a_2, \ldots)$ is a sequence in V and $\sum_{i\in\mathbb{Z}_{>0}} \|a_i\|$ converges.

To show: $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

Let
$$s_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad S_n = \sum_{i=1}^{n} \|a_i\|$$

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Since the sequence $(S_1, S_2, \ldots)$ converges the sequence $(S_1, S_2, \ldots)$ is Cauchy.

Since

$$\|s_n - s_m\| = \|\sum_{i=m+1}^{n} a_i\| \leq \sum_{i=m+1}^{n} \|a_i\| = \|S_n - S_m\|.$$

then the sequence $(s_1, s_2, \ldots)$ is Cauchy.

Since V is complete, the sequence $(s_1, s_2, \ldots)$ converges.

So $\sum_{i \in \mathbb{Z}_{>0}} a_i$ converges.

$\Leftarrow$ Assume that V satisfies $(*)$.

To show: V is complete.

Let $(s_1, s_2, \ldots)$ be a Cauchy sequence in V.

~~Using~~ To show: $(s_1, s_2, \ldots)$ converges.

Using that $(s_1, s_2, \ldots)$ is Cauchy,

let $k_n \in \mathbb{Z}_{>0}$ be such that

if $r, m \in \mathbb{Z}_{>k_n}$ then $\|s_r - s_m\| < \frac{1}{2^n}$.

Let $a_1 = s_{k_1}, a_2 = s_{k_2} - s_{k_1}, a_3 = s_{k_3} - s_{k_2}, \ldots$

Then $\|a_n\| < \frac{1}{2^n}$

So $\sum_{n \in \mathbb{Z}_{>0}} \|a_n\| < \sum_{n \in \mathbb{Z}_{>0}} \frac{1}{2^n} = 1.$

So $\sum_{n \in \mathbb{Z}_{>0}} \|a_n\|$ converges.

Since V satisfies $(*)$ then $\sum_{n \in \mathbb{Z}} a_n$ converges.

So the sequence $(s_{k_1}, s_{k_2}, \ldots)$ converges, since
$s_{k_1} = a_1,\ s_{k_2} = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \ldots$

So the sequence $(s_1, s_2, \ldots)$ has a cluster point.

Since $(s_1, s_2, \ldots)$ is Cauchy and has a cluster point then $(s_1, s_2, \ldots)$ converges.

So V is complete. //