Metric and Hilbert Spaces Lecture 1   26.07.2016

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Let X be a set.   Favorite example:

$$X = \{1, 2, 3\}$$

A topology on X is a collection $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{V \in \mathcal{S}} V\right) \in \mathcal{T}$

(c) If $\ell \in \mathbb{Z}_{>0}$ and $V_1, V_2, \ldots, V_\ell \in \mathcal{T}$ then

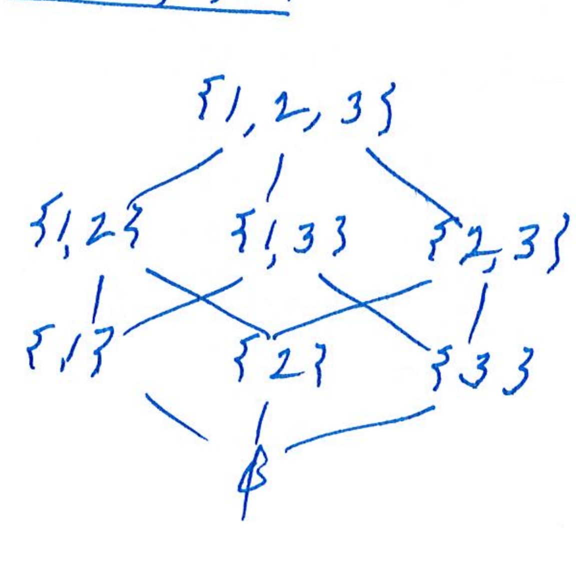
$$(V_1 \cap V_2 \cap \cdots \cap V_\ell) \in \mathcal{T}.$$

In English: (1) $\emptyset$ and X are open

(2) Unions of open sets are open

(3) Finite intersections of open sets are open.

Subsets of $\{1, 2, 3\}$



Topologies on $X = \{1,2,3\}$

$$\mathcal{T}_1 = \left\{ \begin{matrix} X, \\ \phi \end{matrix} \right\}$$

$$\mathcal{T}_2 = \left\{ \begin{matrix} X, \\ \{1\}, \\ \phi \end{matrix} \right\} \qquad \mathcal{T}_3 = \left\{ \begin{matrix} X, \\ \{2\}, \\ \phi, \end{matrix} \right\} \qquad \mathcal{T}_4 = \left\{ \begin{matrix} X, \\ \{3\} \\ \phi \end{matrix} \right\}$$

$$\mathcal{T}_5 = \left\{ \begin{matrix} X, \\ \{1,2\}, \\ \phi \end{matrix} \right\} \qquad \mathcal{T}_6 = \left\{ \begin{matrix} X \\ \{1,3\} \\ \phi \end{matrix} \right\} \qquad \mathcal{T}_7 = \left\{ \begin{matrix} X \\ \{2,3\} \\ \phi \end{matrix} \right\}$$

$$\mathcal{T}_8 = \left\{ \begin{matrix} X, \\ \{1,2\}, \\ \{1\}, \{2\} \\ \phi \end{matrix} \right\} \qquad \mathcal{T}_9 = \left\{ \begin{matrix} X \\ \{1,3\} \\ \{1\} \quad \{3\} \\ \phi \end{matrix} \right\} \qquad \mathcal{T}_{10} = \left\{ \begin{matrix} X \\ \{2,3\} \\ \{2\} \quad \{3\} \\ \phi \end{matrix} \right\}$$

$$\mathcal{T}_{11} = \left\{ \begin{matrix} X \\ \{1,2\} \quad \{1,3\} \\ \{1\} \\ \phi \end{matrix} \right\} \qquad \mathcal{T}_{12} = \left\{ \begin{matrix} X \\ \{1,2\} \quad \{2,3\} \\ \{2\} \\ \phi \end{matrix} \right\} \qquad \mathcal{T}_{13} = \left\{ \begin{matrix} X \\ \{1,3\} \quad \{2,3\} \\ \{3\} \\ \phi \end{matrix} \right\}$$

$$\mathcal{T}_{14} = \left\{ \begin{matrix} X \\ \{1,2\} \quad \{1,3\} \quad \{2,3\} \\ \{1\} \quad \{2\} \quad \{3\} \\ \phi \end{matrix} \right\}$$

Let X be a set.

$$X\times X = \{(x,y) \mid x,y\in X\}$$

is the set of pairs of elements of X.

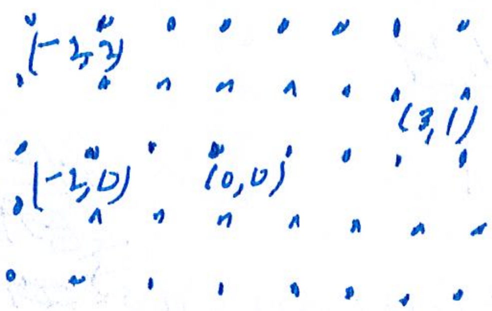
Let $E\subseteq X$. Define

$$\sigma(E) = \{(y,x) \mid (x,y)\in E\}.$$

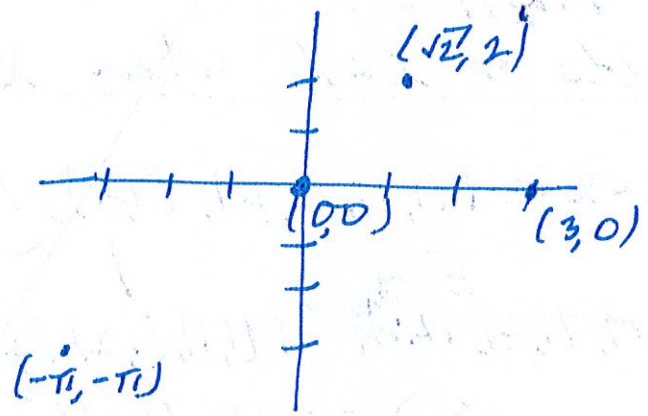
The <u>diagonal of X</u> is

$$\Delta_X = \{(x,x) \mid x\in X\}.$$

<u>Favorite examples of $X\times X$</u>



$\mathbb{Z}\times\mathbb{Z}$



Let $X=\{1,2\}$.

$$X\times X = \{(1,1),(1,2),(2,1),(2,2)\}$$

$$E = \{(1,1),(1,2),(2,2)\} \qquad \sigma(E) = \{(1,1),(2,1),(2,2)\}$$

$$\Delta_X = \{(1,1),(2,2)\}$$

Let X be a set.

A <u>uniformity on X</u> is a collection $\mathcal{U}$ of subsets of $X\times X$ such that

(a) If $E\in\mathcal{U}$ then $\Delta_X\subseteq E\subseteq X\times X$.

(b) If $E\in\mathcal{U}$ then $\sigma(E)\in\mathcal{U}$.

(c) If $E\in\mathcal{U}$ and $B\subseteq X\times X$ and $E\subseteq B$ then $B\in\mathcal{U}$.

(d) If $\ell\in\mathbb{Z}_{>0}$ and $E_1,\ldots,E_\ell\in\mathcal{U}$ then

$$E_1\cap E_2\cap\cdots\cap E_\ell\in\mathcal{U}.$$

(e) If $E\in\mathcal{U}$ then there exists $M\in\mathcal{U}$ with

$$M\times_X M\subseteq E \quad\text{where}$$

$$M\times_X M=\left\{(x,y)\ \middle|\ \begin{array}{l}\text{there exists } z\in X \text{ with}\\ (x,z)\in M \text{ and } (y,z)\in M\end{array}\right\}$$

<u>Favourite example</u>: $X=\{1,2\}$

$$X\times X=\{(1,1)\ (1,2)\ (2,1)\ (2,2)\}$$

$$E=\{(1,1)\ (1,2)\ (2,2)\} \qquad \sigma(E)=\{(1,1),(2,1),(2,2)\}$$

$$\Delta_X=\{(1,1),(2,2)\}$$

The uniformities on X are:

$$\begin{array}{ccc} & \checkmark & \\ \times & & \times \\ & \times & \end{array} \quad\text{and}\quad \begin{array}{ccc} & \checkmark & \\ \checkmark & & \checkmark \\ & \checkmark & \end{array}$$

Let $(X, \mathcal{E})$ be a uniform space.

Let $x \in X$ and $E \in \mathcal{E}$.

The <u>E-neighborhood of x</u> is

$$B_E(x) = \{y \in X \mid (x,y) \in E\}$$

The <u>uniform space topology</u> on X is

$$\mathcal{T} = \left\{ U \subseteq X \;\middle|\; \begin{array}{l} \text{If } x \in U \text{ then there exists} \\ E \in \mathcal{E} \text{ such that } U \supseteq B_E(x) \end{array} \right\}$$

<u>Favourite examples:</u> $X = \{1, 2\}$

$$\begin{array}{ccc} & X \times X & \\ E & & \sigma(E) \\ & \Delta_X & \end{array}$$

where $E = \{(1,1), (1,2), (2,2)\}$

$$B_E(1) = \{1, 2\} = X$$

$$B_{\sigma(E)}(1) = \{1\}.$$

<u>First uniformity</u> $\mathcal{E}_1 = \left\{ \begin{array}{c} X \times X, \\ E, \ \sigma(E) \\ \Delta_X \end{array} \right\}$

has uniform space topology $\mathcal{T}_1 = \left\{ \begin{array}{c} X, \\ \{1\}, \ \{2\}, \\ \emptyset \end{array} \right\}$

<u>Second uniformity</u> $\mathcal{E}_2 = \{ X \times X \}$

has uniform space topology $\mathcal{T}_2 = \left\{ \begin{array}{c} X, \\ \emptyset \end{array} \right\}$