Lecture 19: Metric and Hilbert Spaces

Exercise 4.3.3 (1)

A locally compact space is a topological space $(X, \mathcal{T})$ such that X is Hausdorff and
if $x \in X$ then there exists a neighborhood $N \in \mathcal{N}(x)$ such that N is cover compact.

Isometries

Let (X, d_X) and (Y, d_Y) be metric spaces.
An isometry from X to Y is a function $\varphi: X \to Y$ such that
if $x_1, x_2 \in X$ then $d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2)$.

Completions

Let (X, d) be a metric space.
The completion of (X, d) is a metric space $(\hat{X}, \hat{d})$ with an isometry $\iota: X \to \hat{X}$ such that

(a) $(\hat{X}, \hat{d})$ is complete

(b) $\overline{\iota(X)} = \hat{X}$

Existence of the completion

The completion of (X,d) is

$\hat{X} = \{$Cauchy sequences in $X\}$ with metric

$$d(\vec{x},\vec{y}) = \lim_{n\to\infty} d(x_n,y_n)$$

and with $\quad \iota : X \to \hat{X}$

$$x \mapsto (x, x, \ldots)$$

and Cauchy sequences $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ are equal in $\hat{X}$

$$\vec{x} = \vec{y} \quad \text{if } \lim_{n\to\infty} d(x_n,y_n) = 0.$$

Uniqueness of the completion

Let (X,d) be a metric space.

Let $(\hat{X}_1, \hat{d}_1, \iota_1)$ and $(\hat{X}_2, \hat{d}_2, \iota_2)$ be completions of (X,d). Then there exists a bijective isometry

$$\hat{X}_1 \xrightarrow{\varphi} \hat{X}_2 .$$

HW: If $\varphi: X \to Y$ is an isometry then φ is injective.

Proof Assume $\varphi: X \to Y$ is an isometry.

To show: φ is injective.

To show: If $x_1, x_2 \in X$ and $\varphi(x_1) = \varphi(x_2)$ then $x_1 = x_2$.

Assume $x_1, x_2 \in X$ and $\varphi(x_1) = \varphi(x_2)$.

To show: $x_1 = x_2$.

To show: $d_X(x_1, x_2) = 0$.

$$d_X(x_1, x_2) = d_Y(\varphi(x_1), \varphi(x_2)) = 0.$$

So $x_1 = x_2$.

So φ is injective.

Proof of uniqueness of completion

To show: There exists $\varphi: \hat{X}_1 \to \hat{X}_2$ such that φ is a bijective isometry.

Let $z \in \hat{X}_1$.

Since $\overline{\iota(X)} = \hat{X}_1$ there exists

$(z_1(x_1), z_1(x_2), \ldots)$ in $z(X)$ with $\lim_{n\to\infty} z_1(x_n) = z$.

Define

$$\varphi(z) = \varphi(\lim_{n\to\infty} z_1(x_n)) = \lim_{n\to\infty} \varphi(z_1(x_n))$$

$$= \lim_{n\to\infty} z_2 z_1^{-1} z_1(x_n) = \lim_{n\to\infty} z_2(x_n).$$

To show: (a) φ is an isometry.

(b) $\varphi: \hat{X}_1 \to \hat{X}_2$ is surjective

(b) To show: If $z \in \hat{X}_2$ then there exists $w \in \hat{X}_1$ such that $\varphi(w) = z$.

Assume $z \in \hat{X}_2$.

To show: There exists $w \in \hat{X}_1$ with $\varphi(w) = z$.

Since $z \in \hat{X}_2 = \overline{z_2(X)}$, there exists $(z_2(x_1), z_2(x_2), \ldots)$ with $\lim_{n\to\infty} z_2(x_n) = z$.

Since $(z_2(x_1), z_2(x_2), \ldots)$ converges, $(z_2(x_1), z_2(x_2), \ldots)$ is Cauchy.

Since z_2 is an isometry, $(x_1, x_2, \ldots)$ is Cauchy.

Since z_1 is an isometry $(z_1(x_1), z_1(x_2), \ldots)$ is Cauchy

Since $\hat{X}_1$ is complete, $(z_1(x_1), z_1(x_2), \ldots)$ converges.

Let $w = \lim_{n\to\infty} z_1(x_n)$.

To show: $\varphi(w) = z$.

$$\varphi(w) = \varphi\left(\lim_{n\to\infty} z_1(x_n)\right) = \lim_{n\to\infty} z_2(x_n) = z.$$

So φ is surjective