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Univ. Melbourne ①

Lecture 17: Metric and Hilbert spaces

Sequences of functions   §3.5.3 in the Notes.

Examples: (1) $f_1, f_2, \ldots$ defined by $f_n : [0,1) \to [0,1)$
$$x \mapsto x^n.$$

(2) $f_1, f_2, \ldots$ defined by $f_n : [0,1] \to [0,1]$
$$x \mapsto x^n$$

(3) $f_1, f_2, \ldots$ defined by $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$$x \mapsto x^n$$

Let (X, d_X) and (Y, d_Y) be metric spaces.
Let
$$F = \{\text{functions } f : X \to Y\}$$
and define $d_\infty : F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by
$$d_\infty(f, g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}.$$

Let $f_1, f_2, \ldots$ be a sequence in F. Let $f \in F$.
The sequence $f_1, f_2, \ldots$ <u>converges pointwise</u> to f if $f_1, f_2, \ldots$ satisfies:

if $x \in X$ then $\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0$.

The sequence $f_1, f_2, \ldots$ <u>converges uniformly</u> to f if $f_1, f_2, \ldots$ satisfies:

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

<u>HW</u>: Show that $(f_1, f_2, \ldots)$ converges pointwise to f if and only if $f_1, f_2, \ldots$ satisfies:

if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then $d_Y(f_n(x), f(x)) < \varepsilon$.

<u>HW</u>: Show that $(f_1, f_2, \ldots)$ converges uniformly to f if and only if $f_1, f_2, \ldots$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $x \in X$ and $n \in \mathbb{Z}_{\geq N}$ then $d_Y(f_n(x), f(x)) < \varepsilon$

See www.ms.unimelb.edu.au/~ram/Teaching/2014MetricandHilbert/Ass/SDLNSnos5to10.pdf.

Topologically equivalent metrics §2.11.4

Let X be a set and let

$$d_1 : X \times X \to \mathbb{R}_{\geq 0} \quad \text{and} \quad d_2 : X \times X \to \mathbb{R}_{\geq 0}$$

be metrics on X. Let

$\mathcal{T}_1$ be the metric space topology of (X, d_1)

$\mathcal{T}_2$ the metric space topology of (X, d_2).

The metrics d_1 and d_2 are topologically equivalent if $\mathcal{T}_1 = \mathcal{T}_2$.

The metrics d_1 and d_2 are Lipschitz equivalent if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that if $x, y \in X$ then

$$c_1 d_2(x,y) \leq d_1(x,y) \leq c_2 d_2(x,y).$$

Proposition Let X be a set and let d_1 and d_2 be metrics on X.
If d_1 and d_2 are Lipschitz equivalent then d_1 and d_2 are topologically equivalent.

<u>Proposition</u> Let (X,d) be a metric space. Define $\delta : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\delta(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Then

(a) δ is a metric on X

(b) The metrics d and δ are topologically equivalent.

(c) (X,δ) is bounded.

<u>Example</u> $X = \mathbb{R}$ with

$$d(x,y) = |y-x| \quad \text{and} \quad \delta(x,y) = \frac{|y-x|}{1+|y-x|}$$

Then (X,δ) is bounded,

(X,d) is not bounded, and

(X,δ) and (X,d) have the same metric space topology.