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Lecture 16: Metric and Hilbert spaces Univ. Melbourne

Connected components

Let $(X, \mathcal{T})$ be a topological space. Let $E \subseteq X$.

The set E is <u>connected</u> if there do not exist

open sets A and B in X such that

$A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, $E \subseteq A \cup B$, and

$(A \cap E) \cap (B \cap E) = \emptyset$.

Define an equivalence relation on X by

$x \sim y$ if there exists a connected $E \subseteq X$

with $x \in E$ and $y \in E$.

<u>HW</u>: Show that $\sim$ is an equivalence relation.

<u>HW</u>: Let $x \in X$. Show that the connected component of X which contains x is equal to

$$C_x = \left(\bigcup_{\substack{E \subseteq X \\ E \text{ connected} \\ x \in E}} E \right)$$

Let $(X,\mathcal{T})$ be a topological space. Let $E \subseteq X$.

The set E is <u>path connected</u> if E satisfies:

if $p,q \in E$ then there exists a continuous function $f: [0,1] \to E$ with $f(0)=p$ and $f(1)=q$.

<u>HW</u>: Show that if E is path connected then E is connected.

<u>HW</u>: Give an example of a set E which is connected but not path connected

connected $\underset{\not\Rightarrow}{\Leftarrow}$ path connected.

<u>Theorem</u> Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

(a) If E is connected then $f(E)$ is connected.

(b) If E is compact then $f(E)$ is compact.

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<u>Theorem</u> Let $X = \mathbb{R}$ with the standard metric.

(a) $\mathbb{R}$ is a complete metric space.

(b) $E \subseteq \mathbb{R}$ is connected if and only if E is an interval.

(c) $E \subseteq \mathbb{R}$ is compact if and only if E is closed and bounded.

(d) $E \subseteq \mathbb{R}$ is connected and compact if and only if there exist $m, M \in \mathbb{R}$ such that $E = [m, M]$.

<u>Theorem</u> (Intermediate value theorem).
Let $a, b \in \mathbb{R}$ with $a < b$.

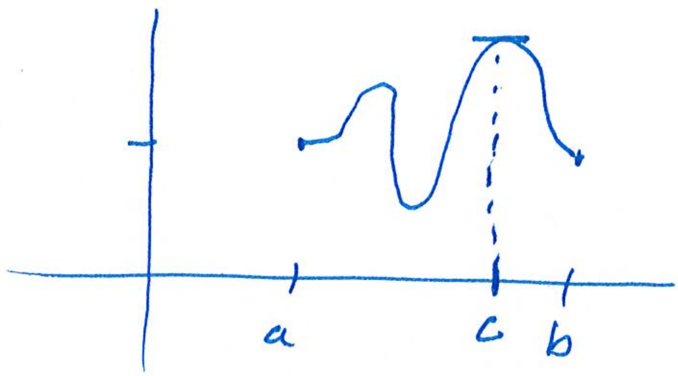
(a) ~~Show~~ If $f : [a,b] \to \mathbb{R}$ is a continuous function and $w \in (f(a), f(b))$ then there exists $c \in (a,b)$ such that $f(c) = w$.

(b) If $f : [a,b] \to \mathbb{R}$ is a continuous function then there exist $m, M \in \mathbb{R}$ such that
$$f([a,b]) = [m, M].$$

<u>Theorem</u> (Rolle's theorem) Let $a, b \in \mathbb{R}$ with $a < b$. If $f : [a,b] \to \mathbb{R}$ is a function such that

f is continuous and $f' : (a,b) \to \mathbb{R}$ exists

and $f(a) = f(b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$.

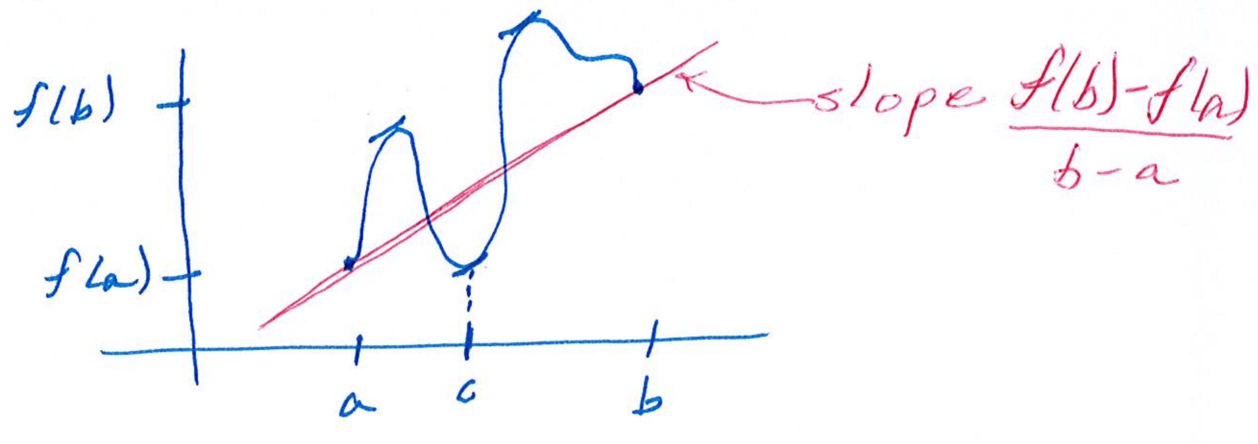


<u>Theorem</u> (Mean value theorem) Let $a, b \in \mathbb{R}$ with $a < b$. If $f : [a,b] \to \mathbb{R}$ is a function such that

f is continuous and $f' : (a,b) \to \mathbb{R}$ exists

then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(c)(b-a)$$



<u>Theorem</u> (Taylor's theorem) Let $a, b \in \mathbb{R}$ with $a < b$. Let $N \in \mathbb{Z}_{\geq 0}$. If $f : [a,b] \to \mathbb{R}$ is a function such that

$f^{(N)} : [a,b] \to \mathbb{R}$ is continuous and

$f^{(N+1)} : (a,b) \to \mathbb{R}$ exists

then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2 + \cdots + \frac{1}{N!} f^{(N)}(a)(b-a)^N + \frac{1}{(N+1)!} f^{(N+1)}(c)(b-a)^{N+1}.$$