Lecture 13: Metric and Hilbert Spaces

Compact spaces

Motivations: In which topological spaces (or metric spaces) do limits always exist?

Example $X = \{0,1\}$ and $(x_1, x_2, \ldots) = (0,1,0,1,\ldots)$ is a sequence with no limit point (if X does not have the topology $\mathcal{T} = \{\emptyset, X\}$). BUT $(0,1,0,1,\ldots)$ does have cluster points.

The main results Let (X,d) be a metric space. Let $A \subseteq X$. Then

$$\begin{tikzcd}
A \text{ is cover compact} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & A \text{ is ball compact} \arrow[r, Rightarrow] & A \text{ is bounded} \\
A \text{ is sequentially compact} \arrow[r, Rightarrow] & A \text{ is Cauchy compact} \arrow[r, Rightarrow] \arrow[ul, phantom, "\Leftarrow \quad +"] & A \text{ is closed in } X.
\end{tikzcd}$$

Definitions

A is sequentially compact if A satisfies: if $(a_1, a_2, a_3, \ldots)$ is a sequence in A then $(a_1, a_2, \ldots)$ has a cluster point in A.

A is Cauchy compact if A satisfies:

if $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then $(a_1, a_2, \ldots)$ has a limit point in A.

A is closed in X if A satisfies:

if $(a_1, a_2, \ldots)$ is a convergent sequence in A then $\lim_{k\to\infty} a_k \in A$.

A is bounded if A satisfies:

there exists $a \in X$ and $M \in \mathbb{R}_{>0}$ such that

$$A \subseteq B_M(a).$$

A is ball compact if A satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in X$ such that

$$A \subseteq B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_\ell)$$

A is cover compact if A satisfies:

if $\mathcal{S} \subseteq \mathcal{T}$ (where $\mathcal{T}$ is the topology on X) and $A \subseteq \left(\bigcup_{S\in\mathcal{S}} S\right)$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $S_1, S_2, \ldots, S_\ell \in \mathcal{S}$ such that

$$A \subseteq S_1 \cup \cdots \cup S_\ell.$$

In English: Every open cover of A has a finite subcover.

Examples

(1) cover compact $\not\Leftarrow$ ball compact.

Let $X = \mathbb{R}$ with the standard metric.

Let $A = (0,1) = \{z \in \mathbb{R} \mid 0 < z < 1\}$.

Then A is ball compact but not cover compact.

Let $\mathcal{S} = \{B_{\varepsilon_x}(x) \mid x \in (0,1)$ where $\varepsilon_x = \min\{\frac{|x|}{2}, \frac{|1-x|}{2}\}\}$.

Then $\mathcal{S}$ is an open cover with no finite subcover.

(2) ball compact $\not\Leftarrow$ bounded.

Let $X = \mathbb{R}$ with the metric given by

$$d(x,y) = \min\{|x-y|, 1\}.$$

Let $A = X$. Then A is bounded but not ball compact.

(3) sequentially compact $\not\Leftarrow$ Cauchy compact

Let $X = \mathbb{R}$ with the standard metric.

Let $A = X$.

(4) Cauchy compact $\not\Leftarrow$ closed in X.

Let $X = (0,1)$ with the standard topology.

Let $A = (0,1)$.

Then $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ is a Cauchy sequence in A which does not converge.

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Proposition Let (X,d) be a metric space
Let $(x_1, x_2, \ldots)$ be a convergent sequence in X.
Then $(x_1, x_2, \ldots)$ is a Cauchy sequence in X.

Proof Let $(x_1, x_2, \ldots)$ be a convergent seq. in X.
Then there exists $z \in X$ such that $\lim_{k\to\infty} x_k = z$.

To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$
such that if $m, n \in \mathbb{Z}_{\geq \ell}$ then $(x_m, x_n) \in B_\varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

The oracle: Since $\lim_{k\to\infty} x_k = z$,
there exists $\ell \in \mathbb{Z}_{>0}$ such that
if $m \in \mathbb{Z}_{\geq \ell}$ then $d(x_m, z) < \frac{\varepsilon}{2}$.

To show: If $m, n \in \mathbb{Z}_{\geq \ell}$ then $(x_m, x_n) \in B_\varepsilon$
Assume $m, n \in \mathbb{Z}_{\geq \ell}$.
To show: $(x_m, x_n) \in B_\varepsilon$
To show: $d(x_m, x_n) < \varepsilon$.

$$d(x_m, x_n) \leq d(x_m, z) + d(z, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(x_m, x_n) \in B_\varepsilon$.
So $(x_1, x_2, \ldots)$ is a Cauchy sequence. //

Proposition Let (X,d) be a metric space. Let $A\subseteq X$. If A is Cauchy compact then A is closed in X.

Proof Assume $A\subseteq X$ and A is Cauchy compact.

To show: A is closed in X.

To show: If $(a_1,a_2,\ldots)$ is a sequence in A and $(a_1,a_2,\ldots)$ converges in X then $\lim_{k\to\infty} a_k\in A$.

Assume $(a_1,a_2,\ldots)$ is a sequence in A and $(a_1,a_2,\ldots)$ converges in X.

By the Proposition that convergent sequences are Cauchy, then $(a_1,a_2,\ldots)$ is a Cauchy sequence.

Since A is Cauchy compact and $(a_1,a_2,\ldots)$ is a Cauchy sequence in A then $(a_1,a_2,\ldots)$ converges in A.

Since limits in metric spaces are unique,

$$z=\lim_{k\to\infty} a_k \in A. //$$