Uniqueness of limits: Hausdorff spaces.

Let $(Y, \mathcal{T}_Y)$ be a topological space.

The topological space $(Y, \mathcal{T}_Y)$ is Hausdorff if $(Y, \mathcal{T}_Y)$ satisfies:

if $z_1, z_2 \in Y$ and $z_1 \neq z_2$ then

there exist $N_1 \in \mathcal{N}(z_1)$ and $N_2 \in \mathcal{N}(z_2)$ with

$$N_1 \cap N_2 = \phi.$$

Proposition Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a function. Assume $(Y, \mathcal{T}_Y)$ is Hausdorff.

(a) Let $a \in X$. If $z_1, z_2 \in Y$

$$\lim_{x \to a} f(x) = z_1 \quad \text{and} \quad \lim_{x \to a} f(x) = z_2$$

then $z_1 = z_2$.

(b) Let $(y_1, y_2, \ldots)$ be a sequence in Y.

If $z_1, z_2 \in Y$ and $\lim_{n \to \infty} y_n = z_1$ and $\lim_{n \to \infty} y_n = z_2$

then $z_1 = z_2$.

Proposition Let (Y, d_Y) be a metric space. Let $\mathcal{T}_Y$ be the metric space topology on Y. Then $(Y, \mathcal{T}_Y)$ is Hausdorff.

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Proof (b) Assume $z_1, z_2 \in Y$ and $\lim_{n\to\infty} y_n = z_1$ and $\lim_{n\to\infty} y_n = z_2$.

To show: $z_1 = z_2$.

Proof by contradiction:

Assume $z_1 \neq z_2$.

Let $N_1 \in \mathcal{N}(z_1)$ and $N_2 \in \mathcal{N}(z_2)$ with $N_1 \cap N_2 = \emptyset$.

~~Then N_1 contains all So~~ there exists $\ell_1 \in \mathbb{Z}_{\geq 0}$ such that $\{y_{\ell_1}, y_{\ell_1+1}, \ldots\} \subseteq N_1$

There exists $\ell_2 \in \mathbb{Z}_{\geq 0}$ such that $\{y_{\ell_2}, y_{\ell_2+1}, \ldots\} \subseteq N_2$

~~Let~~ $\ell = \max\{\ell_1, \ell_2\}$. Then $\{y_\ell, y_{\ell+1}, \ldots\} \subseteq N_1 \cap N_2$.

This is a contradiction to $N_1 \cap N_2 = \emptyset$.

So $z_1 = z_2$.

Proof Let (Y, d_Y) be a metric space.

Let $\mathcal{T}_Y$ be the metric space topology.

To show: $(Y, \mathcal{T}_Y)$ is Hausdorff.

To show: If $z_1, z_2 \in Y$ then there exist $N_1 \in \mathcal{N}(z_1)$ and $N_2 \in \mathcal{N}(z_2)$ with $N_1 \cap N_2 = \emptyset$

Assume $z_1, z_2 \in Y$.

Let $\varepsilon = d(z_1, z_2)$.

Let $N_1 = B_{\frac{\varepsilon}{3}}(z_1)$ and $N_2 = B_{\frac{\varepsilon}{3}}(z_2)$.

To show: $N_1 \cap N_2 = \emptyset$.

Proof by contradiction.

Assume $y \in N_1 \cap N_2$.

Since $N_1 = B_{\frac{\varepsilon}{3}}(z_1)$ and $N_2 = B_{\frac{\varepsilon}{3}}(z_2)$ then

$$d(y, z_1) < \frac{\varepsilon}{3} \quad \text{and} \quad d(y, z_2) < \varepsilon.$$

So

$$\varepsilon = d(z_1, z_2) \leq d(y, z_1) + d(y, z_2) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

This is a contradiction to $\frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

So $N_1 \cap N_2 = \emptyset$.

So $(Y, \mathcal{T}_Y)$ is Hausdorff.

<u>Example of a non Hausdorff space</u>

Let $X = \{0, 1\}$ with $\mathcal{T}_Y = \{\emptyset, Y, \{0\}\}$.

Let $(y_1, y_2, \ldots) = (0, 0, 0, \ldots)$

Then $\lim\limits_{n \to \infty} y_n = 0$ since every $N \in \mathcal{N}(0)$ contains an infinite number of elements of the sequence.

and $\lim\limits_{n \to \infty} y_n = 1$ since every $N \in \mathcal{N}(1)$ (i.e. $N = X$) contains an infinite number of elements of the sequence.

So limits are not unique in $(Y, \mathcal{T}_Y)$.

Some sequence definitions

Let $(Y, \mathcal{T}_Y)$ be a topological space and let $(y_1, y_2, \ldots)$ be a sequence in Y.

Let $z \in Y$.

The point z is a limit point of $(y_1, y_2, \ldots)$ if $(y_1, y_2, \ldots)$ satisfies:

if $N \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $y_n \in N$.

The point z is a cluster point of $(y_1, y_2, \ldots)$ if $(y_1, y_2, \ldots)$ satisfies

if $N \in \mathcal{N}(z)$ and $\ell \in \mathbb{Z}_{>0}$ then there exists $n \in \mathbb{Z}_{\geq \ell}$ such that $y_n \in N$.

Exercise: Show that every ~~clust~~ limit point of $(y_1, y_2, \ldots)$ is a cluster point of $(y_1, y_2, \ldots)$.

Let $(Y, \mathcal{Y})$ be a uniform space. Let $(y_1, y_2, \ldots)$ be a sequence in Y.

The sequence $(y_1, y_2, \ldots)$ is Cauchy if $(y_1, y_2, \ldots)$ satisfies:

if $D \in \mathcal{Y}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq \ell}$ then $(x_m, x_n) \in D$