Lecture 11: Metric and Hilbert spaces

Univ. Melbourne

Closures and limits

Theorem

Let (X, d_X) be a metric space. Let $E \subseteq X$.

Then

$$\bar{E} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } E \\ \text{such that } z = \lim_{n \to \infty} a_n \end{array} \right\}$$

Proof: Let $R = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists } (a_1, a_2, \ldots) \text{ in } E \\ \text{such that } \lim_{n\to\infty} a_n = z \end{array} \right\}$

To show: (a) $R \subseteq \bar{A}$

(b) $\bar{A} \subseteq R$.

(a) To show: If $z \in R$ then $z \in \bar{A}$

Assume $z \in R$

To show: $z \in \bar{A}$

We know there exists $(a_1, a_2, \ldots)$ in A with $\lim_{n\to\infty} a_n = z$

To show: z is a close point to A

To show: If $V \in \mathcal{N}(z)$ then $V \cap A \neq \emptyset$

Assume $V \in \mathcal{N}(z)$

Since $\lim_{n\to\infty} a_n = z$ then V contains all but a finite number of elements of $\{a_1, a_2, \ldots\}$.

Since $\{a_1, a_2, \ldots\} \subseteq A$ then $V \cap A \neq \emptyset$.

So z is a close point to A.

So $z \in \bar{A}$.

(b) To show: $\bar{A} \subseteq R$.

To show: If $z \in \bar{A}$ then $z \in R$.

Assume $z \in \bar{A}$.

To show: $z \in R$.

To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n \to \infty} a_n$.

Using that z is a close point to A,

Let $a_1 \in B_1(z) \cap A$, $a_2 \in B_{\frac{1}{2}}(z) \cap A$, $a_3 \in B_{\frac{1}{3}}(z) \cap A, \ldots$

To show: $z = \lim_{n \to \infty} a_n$

To show: If $N \in \mathcal{N}(z)$ then N contains all but a finite number of $a_1, a_2, \ldots$

Let $N \in \mathcal{N}(z)$.

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(z) \subseteq N$.

Let $\ell \in \mathbb{Z}_{>0}$ with $\frac{1}{\ell} < \varepsilon$.

Then $a_n \in B_{\frac{1}{n}}(z) \subseteq B_{\frac{1}{\ell}}(z) \subseteq B_\varepsilon(z) \subseteq N$, for $n \in \mathbb{Z}_{\geq \ell}$.

So $a_\ell, a_{\ell+1}, \ldots \in N$. So $\lim_{n \to \infty} a_n = z$.

<u>Continuity and limits</u>

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Let $f: X \to Y$ be a function.

The function $f: X \to Y$ is <u>continuous</u> if f satisfies:

If $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.

<u>Recall</u>: $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ has nothing to do with the inverse function to f.

Let $a \in X$. The function $f: X \to Y$ is <u>continuous</u> <u>at a</u> if f satisfies:

if $V \in \mathcal{N}(f(a))$ then $f^{-1}(V) \in \mathcal{N}(a)$.

<u>Theorem</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f: X \to Y$ be a function.

(a) $f: X \to Y$ is continuous if and only if f satisfies: if $a \in X$ then f is continuous at a.

(b) Let $a \in X$. Then $f: X \to Y$ is continuous at a if and only if f satisfies:

$$\lim_{x \to a} f(x) = f(a).$$

<u>Proposition</u> Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f: X \to Y$ be a function.

(a) $f: X \to Y$ is continuous (with respect to the metric space topology on X and Y) if and only if f satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ and $a \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(x), f(a)) < \varepsilon$.

(b) $f: X \to Y$ is continuous if and only if f satisfies:

if $(x_1, x_2, \ldots)$ is a sequence in X and $\lim_{n\to\infty} x_n$ exists

then $f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n)$