Lecture 10: Metric and Hilbert spaces

Univ. Melbourne

Interiors and closures

Let $(X, \mathcal{T})$ be a topological space

An open set in X is $U \in \mathcal{T}$

A closed set in X is C with $C^c \in \mathcal{T}$.

Let $E \subseteq X$.

The interior of E is the subset E° of X such that

(a) E° is open in X and $E^\circ \subseteq E$,

(b) If U is open in X and $U \subseteq E$ then $U \subseteq E^\circ$.

In English: E° is the largest open set contained in E.

The closure of E is the subset $\overline{E}$ of X such that

(a) $\overline{E}$ is closed in X and $\overline{E} \supseteq E$,

(b) If C is closed in X and $C \supseteq E$ then $C \supseteq \overline{E}$.

In English: $\overline{E}$ is the smallest closed set containing E.

<u>Neighborhoods</u> Let $x \in X$.

A <u>neighborhood of x</u> is a subset N of X such that

there exists $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq N$.

The <u>neighborhood filter of x</u> is

$$\mathcal{N}(x) = \{\text{neighborhoods of } x\}$$

<u>Interior points and close points</u>

Let $E \subseteq X$.

An <u>interior point of E</u> is an element $x \in X$ such that

~~if $N \in \mathcal{N}(x)$ then $N \cap E = N \subseteq$~~

there exists $N \in \mathcal{N}(x)$ such that $N \subseteq E$.

A <u>close point to E</u> is an element $x \in X$ such that

if $N \in \mathcal{N}(x)$ then $N \cap E \neq \emptyset$.

<u>Theorem</u> Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$.

(a) The interior of E is the set of interior points of E.

(b) The closure of E is the set of close points of E.

<u>Proof of (a)</u> To show: $E^\circ = \{\text{interior points of } E\}$.

Let $I = \{\text{interior points of } E\}$.

To show: (aa) $I \subseteq E^\circ$

(ab) $E^\circ \subseteq I$

(aa) Let $x \in I$.

Then there exists $N \in \mathcal{N}(x)$ with $N \subseteq E$.

So there exists $U \in \mathcal{T}$ with $x \in U \subseteq N \subseteq E$

Since $U \subseteq E$ and U is open then $U \subseteq E^\circ$.

So $x \in E^\circ$.

So $I \subseteq E^\circ$.

(ab) To show: $E^\circ \subseteq I$.

Let $x \in E^\circ$

Then E° is open and $x \in E^\circ \subseteq E$.

So x is an interior point of E.

So $x \in I$. So $E^\circ \subseteq I$. Thus $E^\circ = I$. //

<u>Limits of functions in topological spaces</u>

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces,

$f: X \to Y$ a function

Let $a \in X$ and $y \in Y$. Write

$\lim_{x \to a} f(a) = y$ if f satisfies

if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

<u>Limits of sequences in topological spaces</u>

Let $(x_1, x_2, \ldots)$ be a sequence in X. Let $y \in X$.

Write

$\lim_{n \to \infty} x_n = y$ if $(x_1, x_2, \ldots)$ satisfies

if $N \in \mathcal{N}(y)$ then N contains all but a finite number of elements of $\{x_1, x_2, \ldots\}$.

More precisely,

$\lim_{n \to \infty} x_n = y$ if $(x_1, x_2, \ldots)$ satisfies:

if $N \in \mathcal{N}(y)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $N \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.

Limits of functions in metric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces

$f: X \to Y$ a function

Let $a \in X$ and $y \in Y$. Write

$\lim_{x \to a} f(a) = y$ if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d_X(x, a) < \delta$ then $d_Y(f(x), y) < \varepsilon$.

Limits of sequences in metric spaces

Let $(x_1, x_2, \ldots)$ be a sequence in X. Let $y \in X$

Write

$\lim_{n \to \infty} x_n = y$ if $(x_1, x_2, \ldots)$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(x_n, y) < \varepsilon$.