(7b) Let $a_n = e^{inx}$ for $n \in \mathbb{Z}$.

To show: (ba) If $n \in \mathbb{Z}$ then $\langle a_n, a_n \rangle = 1$.

(bb) If $m, n \in \mathbb{Z}$ and $m \neq n$ then $\langle a_n, a_m \rangle = 0$.

(ba) Assume $n \in \mathbb{Z}$. To show $\langle a_n, a_n \rangle = 1$.

Then

$$\langle a_n, a_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,\overline{e^{inx}}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = \frac{1}{2\pi} x\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}(\pi - (-\pi)) = \frac{2\pi}{2\pi} = 1.$$

(bb) Assume $m, n \in \mathbb{Z}$ and $m \neq n$.

To show: $\langle a_m, a_n \rangle = 0$.

$$\langle a_m, a_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}\,\overline{e^{inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \frac{1}{2\pi}\cdot\frac{1}{i(m-n)} e^{i(m-n)x}\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi i(m-n)}\left(e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right)$$

$$= \frac{e^{-i(m-n)\pi}}{2\pi i(m-n)}\left(e^{2\pi i(m-n)} - 1\right) = \frac{e^{-i(m-n)\pi}}{2\pi i(m-n)}(1-1) = 0.$$

So $(a_0, a_1, a_{-1}, a_2, a_{-2}, \ldots)$ is an orthonormal sequence.

(7c) To expand x^2 in terms of the a_n, compute $\langle x^2, a_n\rangle$ to get $x^2 = \sum_{n\in\mathbb{Z}} \langle x^2, a_n\rangle a_n$.

$$\langle x^2, a_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \overline{e^{inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx$$

Since

$$\frac{d}{dx}\left(x^2 e^{-inx}\right) = (-in)x^2 e^{-inx} + 2x e^{-inx}$$

then

$$\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \frac{1}{-in} x^2 e^{-inx}\Big]_{-\pi}^{\pi} - \frac{1}{-in}\int_{-\pi}^{\pi} 2x e^{-inx}\,dx$$

So, when $n \neq 0$,

$$\langle x^2, a_n\rangle = \frac{1}{2\pi}\cdot\frac{1}{-in}\left(x^2 e^{-inx}\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x e^{-inx}\,dx\right)$$

$$= \frac{i}{2\pi n}\left(\left(\pi^2 e^{-in\pi} - \pi^2 e^{in\pi}\right) - \int_{-\pi}^{\pi} 2x e^{-inx}\,dx\right)$$

$$= \frac{-i}{2\pi n}\int_{-\pi}^{\pi} 2x e^{-inx}\,dx.$$

Since

$$\frac{d}{dx}\left(2x e^{-inx}\right) = (-in)2x e^{-inx} + 2e^{-inx}$$

then, when $n \neq 0$,

$$\int_{-\pi}^{\pi} 2x e^{-inx}\,dx = \frac{1}{-in} 2x e^{-inx}\Big]_{-\pi}^{\pi} - \left(\frac{1}{-in}\right)\int_{-\pi}^{\pi} 2e^{-inx}\,dx$$

$$= \frac{1}{-in}\left(2\pi e^{-in\pi} + 2\pi e^{in\pi}\right) + \frac{1}{in}\left(\frac{2e^{-inx}}{-in}\Big]_{-\pi}^{\pi}\right)$$

$$= \frac{2\pi i e^{in\pi}}{n}\cdot 2 + \frac{2}{n^2}\left(e^{-in\pi} - e^{in\pi}\right) = \frac{4\pi i}{n} e^{in\pi}$$

Thus, when $n \neq 0$,

$$\langle x^2, a_n\rangle = \frac{-i}{2\pi n}\int_{-\pi}^{\pi} 2x e^{-inx}\,dx = \frac{-i}{2\pi n}\left(\frac{4\pi i}{n} e^{in\pi}\right) = \frac{2}{n^2} e^{in\pi}.$$

When $n = 0$ then

$$\langle x^2, a_0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\frac{x^3}{3}\Big]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{(\pi)^3}{3} - \frac{(-\pi)^3}{3}\right) = \frac{\pi^2}{3}.$$

So

$$x^2 = \langle x^2, a_0\rangle + \sum_{n \in \mathbb{Z}_{>0}} \left(\langle x^2, a_n\rangle a_n + \langle x^2, a_{-n}\rangle a_{-n}\right)$$

$$= \frac{\pi^2}{3} + \sum_{n \in \mathbb{Z}_{>0}} \frac{2}{n^2} e^{in\pi} e^{inx} + \frac{2}{n^2} e^{-in\pi} e^{-inx}$$

(7d) Evaluate at $x=\pi$ to get

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n\in\mathbb{Z}_{>0}} \frac{2}{n^2} e^{in\pi} e^{in\pi} + \frac{2}{n^2} e^{-in\pi} e^{-in\pi}$$

$$= \frac{\pi^2}{3} + \sum_{n\in\mathbb{Z}_{>0}} \frac{4}{n^2}.$$

So $$\sum_{n\in\mathbb{Z}_{>0}} \frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \frac{\pi^2}{6}.$$