Metric and Hilbert Spaces, Assignment 2 Question 6 (6a)

Solutions 2016.

(6) Let H be a Hilbert space.

Let $(a_1, a_2, \dots)$ be an orthonormal sequence in H.

(a) Let $x \in H$.

To show: $\displaystyle\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^2 \le \|x\|^2$.

To show: $\displaystyle\lim_{k\to\infty} \left(\sum_{n=1}^{k} |\langle x, a_n\rangle|^2\right) \le \|x\|^2$.

Assume $k \in \mathbb{Z}_{>0}$ and let

$$x_k = \sum_{n=1}^{k} \langle x, a_n\rangle a_n. \quad \text{Then}$$

$$\|x_k\|^2 = \sum_{n=1}^{k} \langle x, a_n\rangle\langle a_n, x\rangle = \sum_{n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_n\rangle}$$

$$= \sum_{n=1}^{k} |\langle x, a_n\rangle|^2.$$

To show: $\|x_k\|^2 \le \|x\|^2$.

Since

$$\langle x - x_k, x_k\rangle = \langle x, x_k\rangle - \langle x_k, x_k\rangle$$

$$= \sum_{n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_n\rangle} - \sum_{n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_n\rangle}$$

$$= 0,$$

then

$$\|x\|^2 = \langle x, x\rangle = \langle x_k + (x-x_k), x_k - (x-x_k)\rangle$$

$$= \langle x_k, x_k\rangle + \langle x_k, (x-x_k)\rangle + \langle (x-x_k), x_k\rangle + \langle x-x_k, x-x_k\rangle$$

$$= \|x_k\|^2 + 0 + 0 + \|x-x_k\|^2.$$

So $\|x\|^2 \geqslant \|x_k\|^2$.

So $\lim\limits_{k\to\infty} \|x_k\|^2 \le \|x\|^2$

So $\lim\limits_{k\to\infty} \left(\sum\limits_{n=1}^{k} |\langle x, a_n\rangle|^2\right) \le \|x\|^2$

So $\sum\limits_{n=1}^{\infty} |\langle x, a_n\rangle|^2 \le \|x\|^2.$

(6b) Assume $x \in H$.

To show: $P(x) = \sum\limits_{n=1}^{\infty} \langle x, a_n\rangle a_n$ exists in H.

Let
$$x_k = \sum_{n=1}^{k} \langle x, a_n\rangle a_n.$$

To show: $\lim\limits_{k\to\infty} x_k$ exists in H.

Since H is complete,

To show: ~~lim~~ $(x_1, x_2, \ldots)$ is a Cauchy seq. in H.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$.

By Bessel's inequality we know that

$(\|x_1\|, \|x_2\|, \ldots)$ is an increasing sequence in $\mathbb{R}_{\geq 0}$ bounded by $\|x\|$.

So $(\|x_1\|, \|x_2\|, \ldots)$ converges in $\mathbb{R}_{\geq 0}$.

Let $y = \lim_{k \to \infty} \|x_k\|$ and

let $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then

$$\left| y^2 - \|x_k\|^2 \right| < \frac{\varepsilon}{2}.$$

To show: If $r, s \in \mathbb{Z}_{\geq N}$ then $\|x_r - x_s\| < \varepsilon$.

Assume $r, s \in \mathbb{Z}_{\geq N}$.

To show: $\|x_r - x_s\| < \varepsilon$.

If $r < s$

$$\|x_r - x_s\| = \left\| \sum_{j=1}^{r} \langle x, a_j \rangle a_j - \sum_{j=1}^{s} \langle x, a_j \rangle a_j \right\|^2$$

(6b) and (6c).

$$= \left\| \sum_{j=r+1}^{s} \langle x, a_j \rangle a_j \right\|^2 = \sum_{j=r+1}^{s} \langle x, a_j \rangle^2$$

$$= \left| \|x_s\|^2 - \|x_r\|^2 \right| = \left| \|x_s\|^2 - y^2 + y^2 - \|x_r\|^2 \right|$$

$$\leq \left| \|x_s\|^2 - y^2 \right| + \left| y^2 - \|x_r\|^2 \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(x_1, x_2, \ldots)$ is a Cauchy sequence in H.

So $\lim\limits_{k \to \infty} x_k$ exists in H.

So $\sum\limits_{j=1}^{\infty} \langle x, a_j \rangle a_j$ exists in H.

So $P(x)$ exists in H.

(6c) Let $W = \text{span}\{a_1, a_2, \ldots\}$.

To show: $P(x) \in \overline{W}$.

Since

$$x_k = \sum_{j=1}^{k} \langle x, a_j \rangle a_j \text{ is an element of } W,$$

and $\lim\limits_{k \to \infty} x_k$ exists in H then
then
$$P(x) = \lim_{k \to \infty} x_k \in \overline{W}.$$

(6d) To show: $x - P(x) \in \overline{W}^{\perp}$

To show: If $b \in \overline{W}$ then $\langle x-P(x), b\rangle = 0$.

Assume $b \in \overline{W}$

Let $(b_1, b_2, \ldots)$ be a sequence in W with $\lim\limits_{n\to\infty} b_n = b$.

To show: $\langle x - P(x), b\rangle = 0$.

Using that $\langle x-P(x), \cdot\rangle : H \to \mathbb{C}$ is continuous,

$$\langle x-P(x), b\rangle = \langle x-P(x), \lim_{n\to\infty} b_n\rangle$$

$$= \lim_{n\to\infty} \langle x-P(x), b_n\rangle.$$

Let $n \in \mathbb{Z}_{>0}$.

Since $b_n \in W$ there exists $\ell \in \mathbb{Z}_{>0}$ and $c_1, \ldots, c_\ell \in \mathbb{C}$ such that

$$b_n = \sum_{r=1}^{\ell} c_r a_r \text{ and}$$

$$\langle x - P(x), b_n\rangle = \sum_{r=1}^{\ell} \bar{c}_r \langle x-P(x), a_r\rangle$$

Using that $\langle \cdot, a_r\rangle : H \to \mathbb{C}$ is continuous and that $\langle x_k, a_r\rangle = \langle x, a_r\rangle$ for $k \in \mathbb{Z}_{\geq r}$ then

$$\langle x-P(x), a_r\rangle = \langle x, a_r\rangle - \langle P(x), a_r\rangle$$

$$= \langle x, a_r\rangle - \langle \lim_{k\to\infty} x_k, a_r\rangle$$

$$= \langle x, a_r \rangle - \lim_{k \to \infty} \langle x, a_r \rangle = \langle x, a_r \rangle - \langle x, a_r \rangle$$

$$= 0.$$

So $\langle x - P(x), b_n \rangle = \sum_{j=1}^{\ell} \bar{c}_j \langle x - P(x), a_j \rangle = 0.$

So $\langle x - P(x), b \rangle = \lim_{n \to \infty} \langle x - P(x), b_n \rangle = \lim_{n \to \infty} 0 = 0.$

So $x - P(x) \in \overline{W}^{\perp}.$