(5a) Let X be a normed vector space and let $T: X \to X$ be a bounded linear operator.

The operator $T: X \to X$ is <u>compact</u> if T satisfies:

If $(x_1, x_2, \ldots)$ is a sequence in $\{x \in H \mid \|x\| = 1\}$ then $(Tx_1, Tx_2, \ldots)$ has a cluster point in X.

(5c) Let $T: H \to H$ be a compact bounded linear operator. Assume $\lambda \in \mathbb{C}$ and $\lambda \neq 0$. Let

$$X_\lambda = \{v \in H \mid Tv = \lambda v\}.$$

To show: (a) X_λ is a subspace of H.

(b) $\dim(X_\lambda)$ is finite.

(a) To show: (aa) If $v_1, v_2 \in X_\lambda$ then $v_1 + v_2 \in X_\lambda$.

(ab) If $v \in X_\lambda$ and $c \in \mathbb{C}$ then $cv \in X_\lambda$.

(aa) Assume $v_1, v_2 \in X_\lambda$.

Then

$$T(v_1+v_2) = Tv_1 + Tv_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1+v_2).$$

So $v_1 + v_2 \in X_\lambda$.

(ab) Assume $v \in X_\lambda$ and $c \in \mathbb{C}$.

To show: $cv \in X_\lambda$.

To show: $T(cv) = \lambda cv$.

$T(cv) = cT(v) = c\lambda v = \lambda(cv)$.

So $cv \in X_\lambda$.

So X_λ is a subspace.

(b) To show: $\dim(X_\lambda)$ is finite.

Proof by contradiction.

Assume X_λ is infinite dimensional.

Let $(e_1, e_2, \ldots)$ be an orthonormal sequence in in X_λ.

Then, if $m, n \in \mathbb{Z}_{>0}$ and $m \neq n$, then

$$\|Te_m - Te_n\|^2 = \|\lambda e_m - \lambda e_n\|^2 = |\lambda|^2 \|e_m - e_n\|^2$$

$$= |\lambda|^2 \langle e_m - e_n, e_m - e_n \rangle$$

$$= |\lambda|^2 (1 - 0 - 0 + 1) = 2|\lambda|^2.$$

So $d(Te_m, Te_n) = |\lambda| \cdot \sqrt{2}$.

So $(Te_1, Te_2, \ldots)$ does not have a cluster point in H.

So T is not compact.

This is a contradiction to T is compact.

So $\dim(X_\lambda)$ is finite.

(5b) Let $T: \ell^2 \to \ell^2$ be given by

$$T(a_1, a_2, \ldots) = (a_1, 0, 0, \ldots)$$

and let $S: \ell^2 \to \ell^2$ be the identity operator:

$$S(a_1, a_2, \ldots) = (a_1, a_2, \ldots).$$

To show: (a) T is compact

(b) S is not compact

(b) To show: S is not compact.

To show: There exists a sequence $(u_1, u_2, \ldots)$ with $\|u_i\| = 1$ such that $(Su_1, Su_2, \ldots)$ does not have a cluster point

Let $(u_1, u_2, \ldots) = (e_1, e_2, \ldots)$ where

$e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots)$, ...

Then $(Se_1, Se_2, \ldots) = (e_1, e_2, \ldots)$ and

since

$$\|e_m - e_n\|^2 = \|e_m\|^2 + \|e_n\|^2 = 1 + 1 = 2$$

when $m \neq n$ then $(Se_1, Se_2, \ldots)$ does not have a cluster point.

So S is not compact.

(a) To show: T is compact.

To show: If $(u_1, u_2, \ldots)$ is a sequence in l^2 with $\|u_i\| = 1$ then $(Tu_1, Tu_2, \ldots)$ has a cluster point.

Let $(u_1, u_2, \ldots)$ be a sequence in l^2,

$$u_1 = (a_{11}, a_{12}, a_{13}, \ldots)$$
$$u_2 = (a_{21}, a_{22}, a_{23}, \ldots)$$
$$\vdots$$

with $\|u_i\| = 1$. Then $|a_{i1}| \le 1$.

Then $(Tu_1, Tu_2, \ldots)$ is given by

$$Tu_1 = (a_{11}, 0, 0, \ldots)$$
$$Tu_2 = (a_{21}, 0, 0, \ldots)$$
$$\vdots$$

and $(a_{11}, a_{21}, \ldots)$ is a sequence in $[-1, 1]$.

Since $[-1, 1]$ is a closed bounded interval in $\mathbb{R}$ then $[-1, 1]$ is compact. So

$(a_{11}, a_{21}, \ldots)$ has a cluster point $z \in [-1, 1]$.

Then

$(z, 0, 0, \ldots)$ is a cluster point of $(Tu_1, Tu_2, \ldots)$.

So T is compact.