(4) Let H be a Hilbert space and let $T: H \to H$ be a bounded self adjoint compact linear operator.

(4a) To show: There exists $x \in H$ with $\|x\| = 1$ and

$$|\langle Tx, x\rangle| = \|T\|.$$

From question (3) we know

$$\|T\| = \sup\{ |\langle Tu, u\rangle| \mid \|u\| = 1\}. \quad \text{Let } \lambda = \|T\|.$$

So there exists a sequence $(u_1, u_2, \ldots)$ in $\{u \in H \mid \|u\| = 1\}$ such that

$$(|\langle Tu_1, u_1\rangle|, |\langle Tu_2, u_2\rangle|, \ldots) \text{ is increasing and}$$

$$\lim_{n\to\infty} |\langle Tu_n, u_n\rangle| = \lambda.$$

Since T is compact $(Tu_1, Tu_2, \ldots)$ has a cluster point y.

Let $(u_{n_1}, u_{n_2}, \ldots)$ be a subsequence of $(u_1, u_2, \ldots)$ such that

$$\lim_{k\to\infty} Tu_{n_k} = y.$$

Let $$x = \frac{y}{\lambda} = \frac{y}{\|T\|}.$$

To show: (aa) $\lim_{k\to\infty} u_{n_k} = x$

(ab) $\|x\| = 1$.

(ac) $|\langle Tx, x\rangle| = \|T\|$

(b) $Tx = \lambda x$.

(aa) To show: $\lim_{k\to\infty} u_{n_k} = x$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then $\|x - u_{n_k}\| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{>0}$ then $\|y - \lambda u_{n_k}\| < \varepsilon$

Using that $\|y - \lambda u_{n_k}\| \leq \|y - Tu_{n_k}\| + \|Tu_{n_k} - \lambda u_{n_k}\|$,

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{>0}$ then $\|y - Tu_{n_k}\| < \frac{\varepsilon}{2}$ and $\|Tu_{n_k} - \lambda u_{n_k}\| < \frac{\varepsilon}{2}$.

To show: (aaa) There exists $N_1 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N_1}$ then $\|Tu_{n_k} - \lambda u_{n_k}\|^2 \leq \varepsilon^2$.

(aab) There exists $N_2 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N_2}$ then $\|y - Tu_{n_k}\| < \frac{\varepsilon}{2}$

(aab) holds since $\lim_{k\to\infty} Tu_{n_k} = y$.

(aaa) To show: There exists $N_1 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N_1}$ then $\|Tu_{n_k} - \lambda u_{n_k}\|^2 < \varepsilon^2$

~~Assume~~ Let $N_1 \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\geq N_1}$ then

$$|\langle Tu_{n_k}, u_{n_k}\rangle| > \lambda - \frac{\varepsilon^2}{2\lambda}$$

To show: If $k \in \mathbb{Z}_{\geq N_1}$ then $\|Tu_{n_k} - \lambda u_{n_k}\| \leq \varepsilon^2$

Assume $k \in \mathbb{Z}_{\geq N_1}$.

To show: $\|Tu_{n_k} - \lambda u_{n_k}\|^2 \leq \varepsilon^2$.

$$\begin{aligned}\|Tu_{n_k} - \lambda u_{n_k}\|^2 &= \langle Tu_{n_k} - \lambda u_{n_k}, Tu_{n_k} - \lambda u_{n_k}\rangle \\ &= \|Tu_{n_k}\|^2 - 2\lambda\langle Tu_{n_k}, u_{n_k}\rangle + \lambda^2\|u_{n_k}\|^2\end{aligned}$$

(using that $\langle Tu_{n_k}, u_{n_k}\rangle = \langle u_{n_k}, Tu_{n_k}\rangle = \overline{\langle Tu_{n_k}, u_{n_k}\rangle}$ since T is self adjoint).

So
$$\begin{aligned}\|Tu_{n_k} - \lambda u_{n_k}\|^2 &= \|Tu_{n_k}\|^2 - 2\lambda\langle Tu_{n_k}, u_{n_k}\rangle + \lambda^2\|u_{n_k}\|^2 \\ &\leq \|T\|^2 - 2\lambda\langle Tu_{n_k}, u_{n_k}\rangle + \lambda^2 \\ &< 2\lambda^2 - 2\lambda\left(\lambda - \frac{\varepsilon^2}{2\lambda}\right) = \varepsilon^2.\end{aligned}$$

So $\lim_{k\to\infty} (T-\lambda)u_{n_k} = 0$.

Combining (aaa) and (aab),

Let $N = \max\{N_1, N_2\}$ so that

if $k \in \mathbb{Z}_{\geq N}$ then

$$\|y - \lambda u_{n_k}\| \leq \|y - Tu_{n_k}\| + \|Tu_{n_k} - \lambda u_{n_k}\| < \varepsilon.$$

So. $\lim\limits_{k\to\infty} \lambda u_{n_k} = y.$

So $\lim\limits_{k\to\infty} u_{n_k} = x.$

(ab) To show: $\|x\| = 1.$

$$\|x\| = \left\|\frac{y}{\lambda}\right\| = \lim_{k\to\infty} \left\|\frac{\lambda u_{n_k}}{\lambda}\right\| = \lim_{k\to\infty} \|u_{n_k}\| = 1,$$

where we have used that $\| \; \| : H \to \mathbb{R}_{\geq 0}$ is continuous to conclude that $\|y\| = \|\lim\limits_{k\to\infty} \lambda u_{n_k}\|$
$= \lim\limits_{k\to\infty} \|\lambda u_{n_k}\|.$

~~(ac) To show: $|\langle Tx, x\rangle| = \|T\|.$~~

(b) To show: $Tx = \lambda x.$

To show: $\|Tx - \lambda x\| = 0.$

To show: $\|Ty - \lambda y\| = 0.$

$$\|Ty - \lambda y\| = \|T(\lim_{k\to\infty} \lambda u_{n_k}) - \lambda y\|$$

$$= \|\lim_{k\to\infty} (T\lambda u_{n_k}) - \lambda y\|$$

$$= \lambda \lim_{k\to\infty} \|T u_{n_k} - y\| = 0,$$

where we have used

continuity of T to conclude $\lim_{k\to\infty}(T\lambda u_{n_k}) = T(\lim_{k\to\infty} \lambda u_{n_k})$,

linearity of T to conclude $T\lambda u_{n_k} = \lambda T u_{n_k}$,

and continuity of $\|\ \|$ to conclude,

$$\|\lim_{k\to\infty} T\lambda u_{n_k} - \lambda y\| = \lim_{k\to\infty} \|T\lambda u_{n_k} - \lambda y\|,$$

Thus $Ty = \lambda y$.

So $Tx = \lambda x$.

(ac) To show: $|\langle Tx, x\rangle| = \|T\|$

$$|\langle Tx, x\rangle| = |\langle \lambda x, x\rangle| = |\lambda \langle x, x\rangle|$$

$$= |\lambda|\,\|x\| = |\lambda| = \lambda,$$

where the last equality follows from $\lambda = \|T\| \in \mathbb{R}_{\geq 0}$.

(4c) The goal is to find an eigenvector of

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & \pi \\ -2 & \pi & 0 \end{pmatrix}$$ with eigenvalue $\|T\|$.

by using the method of the proof of (a).
Wolfram alpha says that

the eigenvalues of A are approximately
$\lambda_1 \approx -6.59265$
$\lambda_2 \approx 5.61609$
$\lambda_3 \approx 1.96656$

and the corresponding eigenvectors are approximately

$v_1 \approx (1.15992, -1.35753, 1)$

$v_2 \approx (4.84181, 4.87005, 1)$

$v_3 \approx (-0.509751, 0.301459, 1)$.

Since we have to work with approximations anyway, lets do the question with the approximation

$$\hat{A} = \begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$$ as this will probably make some computations more palatable.

In fact, Wolfram alpha says that

the eigenvalues of A are approximately

$\lambda_1 \simeq -6.49139$

$\lambda_2 \simeq 5.5898$

$\lambda_3 \simeq 1.90158$

with corresponding eigenvectors approximately

$\vec{v}_1 \simeq (1.18423, -1.37431, 1)$

$\vec{v}_2 \simeq (5.82293, 5.74522, 1)$

$\vec{v}_3 \simeq (-0.480849, 0.313295, 1)$.

So, indeed the ~~change from~~ approximation of π as 3 does not change the results too drastically.

Now, part (a) says we should try to maximize $|\langle Tu, u\rangle|$ with $\|u\|=1$.

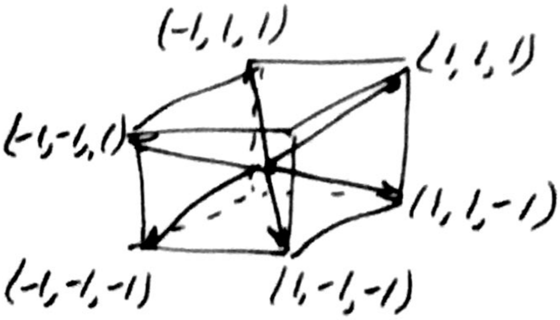
If $u = \frac{x}{\|x\|}$ then

$$\langle Tu, u\rangle = \left\langle T\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle = \frac{\langle Tx, x\rangle}{\|x\|^2}$$

and if $x = (a,b,c)$ then

$$\frac{\langle Tx, x\rangle}{\|x\|^2} = \frac{(a,b,c)\begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & 3 \\ -2 & 3 & 0\end{pmatrix}\begin{pmatrix} a \\ b \\ c\end{pmatrix}}{a^2+b^2+c^2} = \frac{a^2+10ab-4ac+6bc}{a^2+b^2+c^2}.$$

Then, let's try some vectors at different angles to try to maximize this.



Of course,

$$|\langle T(-u), -u\rangle| = |\langle Tu, u\rangle|$$

so there is no point testing both u and $-u$.

If $x=(1,1,1)$ then $\langle Tx,x\rangle = \frac{1+10-4+6}{3} = \frac{13}{3}$

If $x=(1,1,-1)$ then $\langle Tx,x\rangle = \frac{1+10+4-6}{3} = \frac{9}{3}$

If $x=(1,-1,1)$ then $\langle Tx,x\rangle = \frac{1-10-4-6}{3} = -\frac{19}{3} = -6.33$

If $x=(1,-1,-1)$ then $\langle Tx,x\rangle = \frac{1-10+4+6}{3} = \frac{1}{3}$

If $x=(1,0,0)$ then $\langle Tx,x\rangle = \frac{1}{1} = 1$

If $x=(0,1,0)$ then $\langle Tx,x\rangle = \frac{0}{1} = 0$

If $x=(0,0,1)$ then $\langle Tx,x\rangle = \frac{0}{1} = 0$

If $x=(1,1,0)$ then $\langle Tx,x\rangle = \frac{1+10}{2} = \frac{11}{2} = 5.5.$

If $x=(1,0,1)$ then $\langle Tx,x\rangle = \frac{1-4}{2} = -\frac{3}{2}.$

If $x=(0,1,1)$ then $\langle Tx,x\rangle = \frac{6}{2} = 3.$

If $x=(1,-1,0)$ then $\langle Tx,x\rangle = \frac{1-10}{2} = -\frac{9}{2} = -4.5.$

If $x=(1,0,-1)$ then $\langle Tx,x\rangle = \frac{1+4}{2} = \frac{5}{2}$

If $x=(0,1,-1)$ then $\langle Tx,x\rangle = \frac{-6}{2} = -3.$

Indeed $x=(1,-1,1)$ has

$|\langle Tx,x\rangle| = +6.33$ as maximum among these values.

Also

$$\frac{\|Tx\|^2}{\|x\|^2} = \frac{\left\|\begin{pmatrix}1&5&-2\\5&0&3\\-2&3&0\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|^2}{\|x\|^2} = \frac{\|(-6,8,-5)\|^2}{1+1+1} = \frac{36+64+25}{3} = \frac{125}{3} = 41.3$$

So, we'd guess that $x = (1,-1,1)$ is close to an eigenvector and

$$\|T\| \text{ is close to } \sqrt{\frac{\|Tx\|^2}{\|x\|^2}} = \sqrt{41.3} \approx 6.3.$$

This look pretty close to what the theory indicates should happen.

Lets try a bit of power iteration:

$$\begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} -6 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 44 \\ -45 \\ 36 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}\begin{pmatrix} 44 \\ -45 \\ 36 \end{pmatrix} = \begin{pmatrix} \cancel{-1}+176-72 \\ 220+108 \\ -88-135 \end{pmatrix} = \begin{pmatrix} -349 \\ 328 \\ -223 \end{pmatrix}$$

If $(6.3)^3 \approx 6 \cdot 7 \cdot 6 = 6 \cdot 42 = 252$ then

$$\frac{1}{252}\begin{pmatrix} -349 \\ 328 \\ -223 \end{pmatrix} \approx \begin{pmatrix} -1.38492 \\ 1.30158 \\ -.88492 \end{pmatrix}$$

which is proportional to $\begin{pmatrix} 1.565 \\ -1.47 \\ 1 \end{pmatrix}$