(3a) Let V be a normed vector space and let $T: V \to V$ be a linear operator.

The <u>norm</u> of T is

$$\|T\| = \sup\{ \|Tx\| \mid x \in V \text{ and } \|x\| = 1\}.$$

(3b) A <u>bounded</u> <u>linear operator</u> is a linear operator $T: V \to V$ such that

$$\|T\| < \infty.$$

Let H be a Hilbert space. A <u>self adjoint</u> <u>linear operator</u> is a linear operator $T: H \to H$ such that

if $v, w \in H$ then $\langle Tv, w \rangle = \langle v, Tw \rangle$.

(3c) Let H be a Hilbert space and let $T: H \to H$ be a bounded self adjoint linear operator.
Let

$$M = \sup\{ |\langle Tu, u\rangle| \mid \|u\| = 1\}$$

To show: $\|T\| = M$.

To show: (a) $\|T\| \le M$

(b) $\|T\| \ge M$.

(a) Let $x \in H$ with $Tx \neq 0$ and $\|x\| = 1$.

Let $y = \frac{Tx}{\|Tx\|}$. Then

$$\|Tx\| = \frac{\langle Tx, Tx \rangle}{\|Tx\|} = \langle Tx, y \rangle$$

Using that $\langle Tx, y \rangle \in \mathbb{R}$ then $\langle Tx, y \rangle = \langle y, Tx \rangle$

and

$$\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle$$

$$= \langle Tx, x \rangle + \langle Ty, x \rangle + \langle Tx, y \rangle + \langle Ty, y \rangle$$
$$- \langle Tx, x \rangle + \langle Ty, x \rangle + \langle Tx, y \rangle - \langle Ty, y \rangle$$

$$= 2\langle Ty, x \rangle + 2\langle Tx, y \rangle$$

$= 2\langle y, Tx \rangle + 2\langle Tx, y \rangle$, since T is self adjoint

$= 4\langle Tx, y \rangle$, since $\langle Tx, y \rangle \in \mathbb{R}$.

So

$$\|Tx\| = \frac{1}{4}\left(\langle T(x+y), x+y \rangle - \langle T(x-y), (x-y) \rangle\right)$$

$$\leq \frac{1}{4}\left|\langle T(x+y), x+y \rangle - \langle T(x-y), (x-y) \rangle\right|$$

$$\leq \frac{1}{4}\left(|\langle T(x+y), x+y \rangle| + |\langle T(x-y), x-y \rangle|\right)$$

$$= \frac{1}{4}\left(\left|\left\langle \frac{T(x+y)}{\|x+y\|}, \frac{x+y}{\|x+y\|} \right\rangle\right| \|x+y\|^2 + \left|\left\langle \frac{T(x-y)}{\|x-y\|}, \frac{x-y}{\|x-y\|} \right\rangle\right| \|x-y\|^2\right)$$

$$\leq \frac{1}{4}\left(M\|x+y\|^2 + M\|x-y\|^2\right)$$

$$= \frac{1}{4}M\left(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle\right)$$

$$= \frac{1}{4}M\left(\begin{array}{l}\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ \quad + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle\end{array}\right)$$

$$= \frac{M}{4}\left(2\|x\|^2 + 2\|y\|^2\right) = \frac{M}{4}(2+2) = M.$$

So $\|Tx\| \leq M.$

So $\|T\| = \sup\{\|Tx\| \mid x \in H, \|x\| = 1\} \leq M.$

(b) To show: $\|T\| \geq M.$

Assume $u \in H$ and $\|u\| = 1.$

Then, by Cauchy-Schwarz,

$$|\langle Tu, u\rangle| \leq \|Tu\| \cdot \|u\| \leq \|T\| \cdot \|u\| \cdot \|u\| = \|T\|.$$

So $M = \sup\{|\langle Tu, u\rangle| \mid u \in H, \|u\| = 1\} \leq \|T\|.$

So $M = \|T\|.$

So $\|T\| = \sup\{|\langle Tu, u\rangle| \mid u \in H, \|u\| = 1\}.$