(2a) Let $T: V \to V$ be a self adjoint linear operator. Let v be an eigenvector of T with eigenvalue λ.

In order for "eigenvector" to make good sense, we need $v \neq 0$.

To show: $\lambda \in \mathbb{R}$.

To show: $\lambda = \bar{\lambda}$.

$$\lambda \langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle$$

$$= \langle v, Tv\rangle, \text{ since } T \text{ is self adjoint,}$$

$$= \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle.$$

Since $v \neq 0$ then $\langle v, v\rangle \neq 0$ and so $\lambda = \bar{\lambda}$.

So $\lambda \in \mathbb{R}$.

(2b) Let λ and γ be eigenvalues of T with $\gamma \neq \lambda$.

Then

$$X_\lambda = \{v \in V \mid Tv = \lambda v\} \neq \{0\} \text{ and}$$

$$X_\gamma = \{w \in V \mid Tw = \gamma w\} \neq \{0\}.$$

To show: X_λ is orthogonal to X_γ.

To show: If $v \in X_\lambda$ and $w \in X_\gamma$ then $\langle v, w \rangle = 0$.

Assume $v \in X_\lambda$ and $w \in X_\gamma$.

Then

$$\lambda \langle v, w \rangle = \langle \lambda v, w \rangle = \langle Tv, w \rangle$$

$$= \langle v, Tw \rangle, \text{ since } T \text{ is self adjoint}$$

$$= \langle v, \gamma w \rangle = \bar{\gamma} \langle v, w \rangle.$$

So $(\lambda - \bar{\gamma}) \langle v, w \rangle = 0$.

If $(\lambda - \bar{\gamma}) \neq 0$ then $\langle v, w \rangle = 0$.

So the question should be corrected to have $\lambda \neq \bar{\gamma}$, rather than $\lambda \neq \gamma$.

<u>Except</u>, we know, by part (a) that $\lambda \in \mathbb{R}$ and $\gamma \in \mathbb{R}$.

So $\bar{\gamma} = \gamma$ and if $(\lambda - \gamma) \neq 0$ then $\langle v, w \rangle = 0$.

So X_γ is orthogonal to X_λ.