Metric and Hilbert Assignment 1: Solutions.

(5a) The <u>order relation</u> on $\mathbb{R}_{\geq 0}$ is given by

$x \leq y$ if there exists $a \in \mathbb{R}_{\geq 0}$ with $x + a = y$.

If $x, y \in \mathbb{R}_{\geq 0}$ and $x \leq y$ define

$|y - x| = a$, where $a \in \mathbb{R}_{\geq 0}$ and $x + a = y$.

If $x, y \in \mathbb{R}_{\geq 0}$ and $x \geq y$ define

$|y - x| = |x - y|$.

The <u>standard metric</u> on $\mathbb{R}_{\geq 0}$ is the function $d : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $d(x, y) = |y - x|$.

For $\varepsilon \in \mathbb{R}_{>0}$ define

$$B_\varepsilon = \{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mid d(x, y) < \varepsilon\}.$$

The <u>standard uniformity</u> on $\mathbb{R}_{\geq 0}$ is $\mathcal{X} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ where

$E \in \mathcal{X}$ if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $E \supseteq B_\varepsilon$.

(5b) (first part).

A metric space (X, d) is <u>complete</u> if every Cauchy sequence in X converges.

More precisely, (X, d) is <u>complete</u> if (X, d) satisfies:

if $(a_1, a_2, \ldots)$ is a Cauchy sequence in X

then there exists $z \in X$ such that $\lim_{k\to\infty} a_k = z$

A <u>Cauchy sequence</u> in X is a sequence

$(a_1, a_2, \ldots)$ in X such that if $\varepsilon \in \mathbb{R}_{>0}$ then there

exists $\ell \in \mathbb{Z}_{>0}$ such that

if $m, n \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, a_n) < \varepsilon$.

(5b) (second part). To show: $\mathbb{R}$ is complete.

To show: If $(x_1, x_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$

then $(x_1, x_2, \ldots)$ converges.

Assume $(x_1, x_2, \ldots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$.

Each of $x_1, x_2, \ldots$ is a decimal expansion.

To show: $(x_1, x_2, \ldots)$ converges

To show: There exists

$$z = z_\ell \left(\tfrac{1}{10}\right)^\ell + z_{\ell+1}\left(\tfrac{1}{10}\right)^{\ell+1} + \cdots \quad \in \mathbb{R}_{\geq 0}$$

such that $\lim_{n\to\infty} x_n = z$.

Notation: If $a = a_\ell\left(\frac{1}{10}\right)^\ell + a_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots \in \mathbb{R}_{\geq 0}$

let $a_{\leq k} = a_\ell\left(\frac{1}{10}\right)^\ell + a_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots + a_k\left(\frac{1}{10}\right)^\ell$

so that $a_{\leq k} \in \mathbb{Q}_{\geq 0}$ is "the number a to the k^{th} decimal place".

Put $z^{(1)} = (x_1)_{\leq 1+2}$ so that

$$z^{(1)} \in \mathbb{Q}_{\geq 0} \text{ and } d(z^{(1)}, x_1) < \frac{1}{10}$$

(Probably it would have been good enough to put $z^{(1)} = (x_1)_{\leq 1}$ but, to be extriemely safe we put $z^{(1)} = (x_1)_{\leq 1+2}$.)

Put $z^{(2)} = (x_2)_{\leq 2+2}$ so that

$$z^{(2)} \in \mathbb{Q}_{\geq 0} \text{ and } d(z^{(2)}, x_2) < \left(\frac{1}{10}\right)^2$$

Put $z^{(3)} = (x_3)_{\leq 3+2}$ so that

$$z^{(3)} \in \mathbb{Q}_{\geq 0} \text{ and } d(z^{(3)}, x_3) < \left(\frac{1}{10}\right)^3$$

Put $z^{(4)} = (x_4)_{\leq 4+2}$ so that

$$z^{(4)} \in \mathbb{Q}_{\geq 0} \text{ and } d(z^{(4)}, x_4) < \left(\frac{1}{10}\right)^4.$$

For $k \in \mathbb{Z}_{>0}$ put $z^{(k)} = (a_k)_{\leq k+2}$ so that

$$z^{(k)} \in \mathbb{Q}_{\geq 0} \text{ and } d(z^{(k)}, x_k) < \left(\frac{1}{10}\right)^k.$$

This construction makes $z^{(k)}$ so that $z^{(k)} \in \mathbb{Q}_{\geq 0}$ and $z^{(k)}$ is the same as x_k at least up to the k^{th} decimal place.

To show: (a) If $n \in \mathbb{Z}_{\geq 0}$ then there exists $t_n \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq t_n}$ then

$$d(z^{(r)}, z^{(s)}) < \left(\frac{1}{10}\right)^{n+1}$$

This will guarantee that the n^{th} decimal place of $z^{(r)}$ is the same for $r \in \mathbb{Z}_{\geq t_n}$ and we will <u>define</u>

$$z = z_\ell \left(\frac{1}{10}\right)^\ell + z_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots \quad \in \mathbb{R}_{\geq 0}$$

by setting z_n (the n^{th} decimal place of z) to be the same as the n^{th} decimal place of $z^{(r)}$ for $r \in \mathbb{Z}_{\geq t_n}$.

To show: (b) $\lim_{k\to\infty} x_k = z$, so that $(x_1, x_2, \ldots)$ converges to z.

(a) Assume $n \in \mathbb{Z}_{\geq 0}$.

To show: There exists $t_n \in \mathbb{Z}_{\geq 0}$ such that if $r, s \in \mathbb{Z}_{\geq t_n}$ then $d(z^{(r)}, z^{(s)}) < \left(\frac{1}{10}\right)^{n+1}$.

Since $(x_1, x_2, \ldots)$ is a Cauchy sequence we know that there exists $t_n' \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq t_n'}$ then $d(x_r, x_s) < \left(\frac{1}{10}\right)^{n+2}$.

Let $t_n = \max\{t_n', n+2\}$.

To show: If $r, s \in \mathbb{Z}_{\geq t_n}$ then $d(z^{(r)}, z^{(s)}) < \left(\frac{1}{10}\right)^{n+1}$.

Assume $r, s \in \mathbb{Z}_{\geq t_n}$

To show: $d(z^{(r)}, z^{(s)}) < \left(\frac{1}{10}\right)^{n+1}$.

$$
\begin{aligned}
d(z^{(r)}, z^{(s)}) &\leq d(z^{(r)}, x_r) + d(x_r, x_s) + d(x_s, z^{(s)}) \\
&< \left(\frac{1}{10}\right)^{r} + \left(\frac{1}{10}\right)^{n+2} + \left(\frac{1}{10}\right)^{s} \\
&\leq \left(\frac{1}{10}\right)^{n+2} + \left(\frac{1}{10}\right)^{n+2} + \left(\frac{1}{10}\right)^{n+2}, \quad \text{since } r \geq n+2 \text{ and } s \geq n+2 \\
&= 3 \cdot \left(\frac{1}{10}\right)^{n+2} < 10\left(\frac{1}{10}\right)^{n+2} = \left(\frac{1}{10}\right)^{n+1}.
\end{aligned}
$$

To show: (b) With $z = z_\ell \left(\frac{1}{10}\right)^{\ell} + z_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots$ (as defined above so that the nth decimal place of z is the same as the nth decimal place of $z^{(r)}$ for $r > t_n$) then $\lim_{k\to\infty} x_k = z$.

To show: If $r \in \mathbb{Z}_{>0}$ then there exists $s_r \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{>s_r}$ then $d(x_k, z) < \left(\frac{1}{10}\right)^{r}$.

Assume $r \in \mathbb{Z}_{>0}$.

To show: There exists $s_r \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{>s_r}$ then $d(x_k, z) < \left(\frac{1}{10}\right)^{r}$

Let t_r be such that if $k \in \mathbb{Z}_{>t_r}$ then

$$d(z^{(k)}, z) < \left(\frac{1}{10}\right)^{r+1}$$

Let $s_r = \max\{t_r, r+1\}$.

To show: If $k \in \mathbb{Z}_{\geq s_r}$ then $d(x_k, z) < \left(\frac{1}{10}\right)^r$.

Assume $k \in \mathbb{Z}_{\geq s_r}$.

To show: $d(x_k, z) < \left(\frac{1}{10}\right)^r$.

$$\begin{aligned} d(x_k, z) &\leq d(x_k, z^{(k)}) + d(z^{(k)}, z) \\ &< \left(\frac{1}{10}\right)^k + \left(\frac{1}{10}\right)^{r+1} \\ &\leq \left(\frac{1}{10}\right)^{r+1} + \left(\frac{1}{10}\right)^{r+1}, \quad \text{since } k \geq s_r \geq r+1 \\ &= 2\left(\frac{1}{10}\right)^{r+1} < 10 \cdot \left(\frac{1}{10}\right)^{r+1} = \left(\frac{1}{10}\right)^r. \end{aligned}$$

So $\lim_{k \to \infty} x_k = z$.

So $(x_1, x_2, \ldots)$ converges in $\mathbb{R}_{\geq 0}$.