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Metric and Hilbert Assignment 1 Solutions: Question 4 (4a) and (4b)

(4a) Let $a, b \in \mathbb{R}_{\geq 0}$ with $a < b$.

To show: (a,b) is open in $\mathbb{R}_{\geq 0}$.

Using that $a<b$,

Let $x \in \mathbb{R}_{\geq 0}$ such that $a+x=b$

Let $y = \cancel{5\cdot(\frac{1}{10})\cdot x = \frac{x}{2}}\, a + \frac{x}{2} = a + 5\cdot(\frac{1}{10})x$

Then $a = y - \frac{x}{2}$ and $b = y + \frac{x}{2}$ so that

$$(a,b) = (y-\tfrac{x}{2}, y+\tfrac{x}{2}) = B_{\frac{x}{2}}(y).$$

So (a,b) is open in $\mathbb{R}_{\geq 0}$.

(4b) Let $a, b \in \mathbb{R}_{\geq 0}$ with $a<b$.

To show: $[a,b]$ is closed in $\mathbb{R}_{\geq 0}$.

To show: $[a,b]^c = (-\infty, a) \cup (b, \infty)$ is open in $\mathbb{R}_{\geq 0}$.

Since

$$(-\infty,a)\cup(b,\infty) = [0,a) \cup (b, b+2) \cup (b+1, b+3) \cup \cdots$$

$$= [0,a) \cup \Big(\bigcup_{k \in \mathbb{Z}_{\geq 0}} (b+k, b+k+2) \Big)$$

then $(-\infty, a) \cup (b,\infty)$ is a union of open intervals.

So $(-\infty, a)\cup(b,\infty)$ is open.

So $[a,b]$ is closed in $\mathbb{R}_{\geq 0}$.

(4c) A topological space is <u>compact</u> if every open cover has a finite subcover.

More precisely, a topological space $(X,\mathcal{T})$ is <u>compact</u> if $(X,\mathcal{T})$ satisfies:

if $\mathcal{S} \subseteq \mathcal{T}$ such that $X = \left(\bigcup_{s\in\mathcal{S}} s\right)$

then there exists $\ell \in \mathbb{Z}_{>0}$ and $s_1, s_2, \ldots, s_\ell \in \mathcal{S}$ such that $X = s_1 \cup s_2 \cup \cdots \cup s_\ell$.

<u>Second part</u>: Let

$\mathcal{S} = \{[0,2), (1,3), (2,4), \ldots\}$ so that $\mathcal{S}$ is an open cover of $\mathbb{R}_{\geq 0} = [0,2) \cup (1,3) \cup (2,4) \cup \cdots$

Assume $S_{k_1} = (k_1, k_1+2), \ldots, S_{k_\ell} = (k_\ell, k_\ell+2)$ be a finite number of subsets of $\mathcal{S}$.

Let

$$m = \max\{k_1+2, k_2+2, \ldots, k_\ell+2\}.$$

Then $m \in \mathbb{R}_{\geq 0}$ and $m+1 \in \mathbb{R}_{\geq 0}$ and

$$m+1 \notin S_{k_1} \cup S_{k_2} \cup \cdots \cup S_{k_\ell}$$

So $\mathbb{R}_{\geq 0} \neq S_{k_1} \cup S_{k_2} \cup \cdots \cup S_{k_\ell}$.

So $S_{k_1}, \ldots, S_{k_\ell}$ is not a cover of $\mathbb{R}_{\geq 0}$

So $\mathcal{S}$ is an open cover of $\mathbb{R}_{\geq 0}$ with no finite subcover.

So $\mathbb{R}_{\geq 0}$ is not compact

(4d) A topological space $(X, \mathcal{T})$ is locally compact

if $(X, \mathcal{T})$ satisfies:

if $x \in X$ then there exists a neighborhood N of x such that N is cover compact.

(4d) part 2: To show: If $x \in \mathbb{R}_{\geq 0}$ then there exists a neighborhood N of x such that N is cover compact.

Assume $x \in \mathbb{R}_{\geq 0}$

To show: There exists a neighborhood N of x such that N is cover compact.

Let $N = [0, x+1]$.

Since $x \in N$ and $N \supseteq [0, x+1)$ and $[0, x+1)$ is open in $\mathbb{R}_{\geq 0}$, then N is a neighborhood of x.

To show: N is cover compact.

By (4b), $N = [0, x+1]$ is closed in $\mathbb{R}_{\geq 0}$.

Since $N \subseteq B_{2x}(x) = [0, 2x)$ then N is bounded in $\mathbb{R}_{\geq 0}$.

Since N is closed and bounded then, by (6b) of this assignment,

N is cover compact.

So $\mathbb{R}_{\geq 0}$ is locally compact.