2016

Metric and Hilbert Assignment 1 Solutions 23.09.2016 (3a) ①

(3a) Assume $A \subseteq \mathbb{R}_{\geq 0}$ and $A \neq \emptyset$ and A is bounded.

To show: $\sup(A)$ exists.

To show: There exists $z \in \mathbb{R}_{\geq 0}$ such that $\sup(A) = z$.

Let

$$B = \{\gamma \in \mathbb{R}_{\geq 0} \mid \gamma \text{ is an upper bound of } A\}.$$

We know: $B \neq \emptyset$ and $A \neq \emptyset$.

Let $a_1 \in A$ and $b_1 \in B$.

Let $a_{i+1} = a_i$ if $\frac{a_i + b_i}{2} \in B$, and

$a_{i+1} \in A$ with $a_{i+1} > \frac{a_i + b_i}{2}$ if $\frac{a_i + b_i}{2} \notin B$.

Let

$$b_{i+1} = \begin{cases} b_i, & \text{if } \frac{a_i + b_i}{2} \notin B, \\ \frac{a_i + b_i}{2}, & \text{if } \frac{a_i + b_i}{2} \in B. \end{cases}$$

Then $a_i \in A$ and $b_i \in B$ and

$d(a_{i+1}, b_{i+1}) \leq \frac{1}{2} d(a_i, b_i)$ and

$d(a_{i+1}, b_{i+1}) \leq \frac{1}{2^i} d(a_1, b_1)$. and

$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1$

So $(a_1, a_2, \ldots)$ is Cauchy and

$(b_1, b_2, \ldots)$ is Cauchy.

Using that $\mathbb{R}_{\geq 0}$ is complete,

$$z_a = \lim_{k\to\infty} a_k \quad \text{and} \quad z_b = \lim_{k\to\infty} b_k$$

exist in $\mathbb{R}_{\geq 0}$.

To show: (a) If $k \in \mathbb{Z}_{>0}$ then $a_k \leq z_a$

(b) If $k \in \mathbb{Z}_{>0}$ then $z_b \leq b_k$.

(c) $z_a = z_b$

Let $z = z_a = z_b$. (d) If $a \in A$ then $a \leq z$

(e) If $b \in B$ then $z \leq b$.

(a) To show: If $k \in \mathbb{Z}_{>0}$ then $a_k \leq z_a$.

Proof by contradiction.

Assume there exists $k \in \mathbb{Z}_{>0}$ with $a_k > z_a$.

Then $z_a < a_k \leq a_n$ for $n \in \mathbb{Z}_{\geq k}$.

So $d(a_n, z_a) \geq d(z_a, a_k) \neq 0$ for $n \in \mathbb{Z}_{\geq k}$.

This is a contradiction to $\lim_{n\to\infty} a_n = z_a$.

So, if $k \in \mathbb{Z}_{>0}$ then $a_k \leq z_a$.

So there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq \ell}$ then $d(b_n, z) < d(z, a)$.

So $z \leq b_n < a$.

This is a contradiction to $b_n \in B$ being an upper bound of A.

So if $a \in A$ then $a \leq z$.

(e) To show: If $b \in B$ then $z \leq b$.

Proof by contradiction

Assume there exists $b \in B$ with $b < z$.

Then $d(b, z) \in \mathbb{R}_{>0}$

So there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_n, z) < d(b, z)$.

So $b < a_n \leq z$.

This is a contradiction to b being an upper bound of A.

So, if $b \in B$ then $z \leq b$.

By (d) z is an upper bound of A.

By (e) z is a <u>least</u> upper bound of A.

So $z = \sup(A)$.

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(3b) and (3c).

(3b) Let $(a_1, a_2, \ldots) = (1, 2, 3, 4, \ldots)$ in $\mathbb{R}_{\geq 0}$.

To show: $(1, 2, 3, \ldots)$ does not converge in $\mathbb{R}_{\geq 0}$.

Proof by contradiction.

Assume $z \in \mathbb{R}_{\geq 0}$ such that $\lim_{k \to \infty} k = z$.

Let $\varepsilon = \frac{1}{3}$. Since $\lim_{k \to \infty} k = z$ then

there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then $d(z, n) < \frac{1}{3}$.

So $$1 = d(N, N+1) \leq d(N, z) + d(z, N+1)$$
$$< \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

This is a contradiction to $\frac{2}{3} + \frac{1}{3} = 1$ (so $\frac{2}{3} < 1$).

So $(1, 2, 3, \ldots)$ does not converge in $\mathbb{R}_{\geq 0}$.

(3c) Let $(a_1, a_2, \ldots) = (1, 2, 1, 2, 1, 2, \ldots)$ in $\mathbb{R}_{\geq 0}$

~~Let~~ Then $(a_1, a_2, \ldots)$ is bounded since $a_i < 3$ for $i \in \mathbb{Z}_{>0}$.

To show: $(1, 2, 1, 2, \ldots)$ does not converge in $\mathbb{R}_{\geq 0}$.

Proof by contradiction.

Assume $z \in \mathbb{R}_{\geq 0}$ such that $\lim_{k\to\infty} a_k = z$.

Let $\varepsilon = \frac{1}{4}$. Since $\lim_{k\to\infty} a_k = z$ then

there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then $d(z, a_n) < \frac{1}{4}$.

Let $k \in \mathbb{Z}_{\geq N}$ with k odd. Then

$$d(a_k, a_{k+1}) = d(1,2) = 1 \quad \text{and}$$

$$\begin{aligned} d(a_k, a_{k+1}) &\leq d(a_k, z) + d(z, a_{k+1}) \\ &< \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}. \end{aligned}$$

So $1 < \frac{1}{2}$.

This is a contradiction to $\frac{1}{2} + \frac{1}{2} = 1$ (so that $\frac{1}{2} < 1$).

So $(1, 2, 1, 2, \ldots)$ does not converge in $\mathbb{R}_{\geq 0}$.

(3d) Assume $(a_1, a_2, \ldots)$ is a bounded increasing sequence in $\mathbb{R}_{\geq 0}$.

To show: $(a_1, a_2, \ldots)$ converges in $\mathbb{R}_{\geq 0}$.

To show: There exists $z \in \mathbb{R}_{\geq 0}$ such that $\lim\limits_{k \to \infty} a_k = z$.

Since $(a_1, a_2, \ldots)$ is bounded $A = \{a_1, a_2, \ldots\}$ is bounded in $\mathbb{R}_{\geq 0}$.

So $\sup(A)$ exists, by part (a).

Let $z = \sup(A)$.

To show: $\lim\limits_{k \to \infty} a_k = z$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{\geq 0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(a_n, z) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show there exists $N \in \mathbb{Z}_{\geq 0}$ such that if $n \in \mathbb{Z}_{\geq 0}$ then $d(a_n, z) < \varepsilon$.

Since z is a least upper bound of A, $z - \varepsilon$ is not an upper bound of A.

So there exists $a_N \in A$ with $a_N > z - \varepsilon$.

To show: If $n \in \mathbb{Z}_{\geq N}$ then $d(a_n, z) < \varepsilon$.

Assume $n \in \mathbb{Z}_{\geq N}$.

Since $(a_1, a_2, \ldots)$ is increasing $a_N \leq a_n$.

So $z - \varepsilon < a_N \leq a_n \leq z$.

So $0 \leq z - a_n < z - (z - \varepsilon) = \varepsilon$.

So $d(z, a_n) < \varepsilon$.

So $(a_1, a_2, \ldots)$ converges to $\sup(A)$. //

(3e) Let $x_0 + x_1(\frac{1}{10}) + x_2(\frac{1}{10})^2 + \cdots$ be the decimal expansion of $\sqrt{2} = 1.414\ldots$

Let $a_k = x_0 + x_1(\frac{1}{10}) + x_2(\frac{1}{10})^2 + \cdots + x_k(\frac{1}{10})^k$.

Then $(a_1, a_2, a_3, \ldots)$ is an increasing sequence in $\mathbb{Q}_{\geq 0}$ which is bounded by 2.

To show: $(a_1, a_2, a_3, \ldots)$ does not converge in $\mathbb{Q}_{\geq 0}$

Proof by contradiction.

Assume $z \in \mathbb{Q}_{\geq 0}$ and $\lim_{k \to \infty} a_k = z$.

Then, since multiplication is continuous on $\mathbb{Q}_{\geq 0}$.

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$$\lim_{k\to\infty} a_k^2 = \left(\lim_{k\to\infty} a_k\right)^2 = z^2 = 2.$$

So $z \in \mathbb{Q}_{\geq 0}$ and $z^2 = 2$.

Let $z = \frac{p}{q}$ in reduced form.

Then $\frac{p^2}{q^2} = 2$. So $p^2 = 2q^2$.

So p^2 is divisible by 2.

So p is divisible by 2.

So p^2 is divisible by 4.

So q^2 is divisible by 2.

So q is divisible by 2.

So both p and q are divisible by 2.

This is a contradiction to $\frac{p}{q}$ being in reduced form.

So there does not exist $z \in \mathbb{Q}_{\geq 0}$ with $z^2 = 2$.

So there does not exist $z \in \mathbb{Q}_{\geq 0}$ with $\lim_{k\to\infty} a_k = z$.

So $(a_1, a_2, \ldots)$ does not converge in $\mathbb{Q}_{\geq 0}$.