2016 Metric and Hilbert Assignment 1 Solutions: Question 1. 23.09.2016 (1)

(1)(a) The definition of the nonnegative real numbers $\mathbb{R}_{\geq 0}$ is given in the notes on p. 67.

The <u>nonnegative real numbers</u> $\mathbb{R}_{\geq 0}$ is the set

$$\mathbb{R}_{\geq 0} = \left\{ a_\ell \left(\tfrac{1}{10}\right)^\ell + a_{\ell+1}\left(\tfrac{1}{10}\right)^{\ell+1} + \cdots \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z} \text{ and} \\ a_i \in \{0,1,2,\ldots,9\} \end{array} \right\}$$

with

$$x = y \quad \text{if} \quad \lim_{k\to\infty} |x_{\leq k} - y_{\leq k}| = 0,$$

where if $x = x_\ell \left(\tfrac{1}{10}\right)^\ell + x_{\ell+1}\left(\tfrac{1}{10}\right)^{\ell+1} + \cdots$ then $x_{\leq k}$ is x to the kth decimal place,

$$x_{\leq k} = x_\ell \left(\tfrac{1}{10}\right)^\ell + \cdots + x_k \left(\tfrac{1}{10}\right)^k \in \mathbb{Q}_{\geq 0} \quad \text{and}$$

$$\lim_{k\to\infty} |x_{\leq k} - y_{\leq k}| = 0 \quad \text{means that}$$

if $\varepsilon \in \mathbb{Q}_{>0}$ then there exists $N \in \mathbb{Z}_{\geq 0}$ such that if $k \in \mathbb{Z}_{\geq N}$ then $|x_{\leq k} - y_{\leq k}| < \varepsilon$.

The following definitions, for parts (1b), (1c) are taken from p. 67 of the notes and the definition in (1d) is the "proof machine" form of the definition of the standard topology on $\mathbb{R}_{\geq 0}$ given in class.

(1)(b) The <u>addition</u> and <u>multiplication</u> on $\mathbb{R}_{\geq 0}$ are the functions

$$\begin{array}{ccc} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} & \text{and} & \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ (x,y) \longmapsto x+y & & (x,y) \longmapsto xy \end{array}$$

given by

$$\lim_{k\to\infty} |(x+y) - (x_{\leq k} + y_{\leq k})| = 0 \quad \text{and}$$

$$\lim_{k\to\infty} |xy - x_{\leq k} \cdot y_{\leq k}| = 0.$$

(1)(c) The <u>usual order</u> on $\mathbb{R}_{\geq 0}$ is given by

$x \leq y$ if there exists $a \in \mathbb{R}_{\geq 0}$ such that $x+a=y$.

(1)(d) The <u>usual topology</u> on $\mathbb{R}_{\geq 0}$ is given by letting

$\cancel{\mathcal{B} = \{(x-a, x+a) \mid x \in \mathbb{R}_{\geq 0}}$

$$\mathcal{B}_1 = \{(x,y) \mid x \in \mathbb{R}_{\geq 0}, y \in \mathbb{R}_{\geq 0}, x<y\}$$

$$\mathcal{B}_2 = \{(-\infty, y) \mid y \in \mathbb{R}_{\geq 0}\}$$

and ~~setting~~ defining $U \subseteq \mathbb{R}_{\geq 0}$ to be open if there exists $\mathcal{S} \subseteq \mathcal{B}_1 \cup \mathcal{B}_2$ such that

$$U = \Big(\bigcup_{S \in \mathcal{S}} S\Big).$$