# Assignment 1

MAST30026 Metric and Hilbert Spaces
Semester II 2016
Lecturer: Arun Ram
to be turned in before 10am on 8 September 2016

(1) (Definition of the nonnegative real numbers)

(a) Carefully define the nonnegative real numbers $\mathbb{R}_{\geqslant 0}$.
(b) Carefully define the usual addition and multiplication on $\mathbb{R}_{\geqslant 0}$.
(c) Carefully define the usual order on $\mathbb{R}_{\geqslant 0}$.
(d) Carefully define the usual topology $\mathbb{R}_{\geqslant 0}$.

Be careful that your definitions are not circular (i.e. be careful that your definitions are not somehow already using the real numbers to define the real numbers).

(2) (Properties of the order on $\mathbb{R}_{\geqslant 0}$)

(a) Prove that if $a, b, c \in \mathbb{R}_{\geqslant 0}$ and $a \leqslant b$ then $a + c \leqslant b + c$.
(b) Prove that if $x, y \in \mathbb{R}_{\geqslant 0}$ then there exists $n \in \mathbb{Z}_{>0}$ such that $y < nx$.
(c) Prove that if $a, b \in \mathbb{R}_{\geqslant 0}$ and $a < b$ then there exists $c \in \mathbb{Q}_{\geqslant 0}$ (a rational number) such that $a < c < b$.
(d) Prove that if $a, b \in \mathbb{R}_{\geqslant 0}$ and $a < b$ then there exists $c \in (\mathbb{R}_{\geqslant 0} - \mathbb{Q}_{\geqslant 0})$ (an irrational number) such that $a < c < b$.

(3) (Least upper bounds and increasing sequences in $\mathbb{R}_{\geqslant 0}$)

(a) Prove that if $A \subseteq \mathbb{R}_{\geqslant 0}$ and $A \neq \emptyset$ and $A$ is bounded then $\sup(A)$ exists.
(b) Give an example (with proof) of an increasing sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geqslant 0}$ which does not converge.
(c) Give an example (with proof) of a bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geqslant 0}$ which does not converge.
(d) Prove that if $(a_1, a_2, \ldots)$ is an increasing and bounded sequence in $\mathbb{R}_{\geqslant 0}$ then $(a_1, a_2, a_3, \ldots)$ converges.
(e) Give an example (with proof) of an increasing and bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{Q}_{\geqslant 0}$ which does not converge.

(4) (Properties of the topology on $\mathbb{R}_{\geqslant 0}$)

(a) Let $a, b \in \mathbb{R}_{\geqslant 0}$ with $a < b$. Prove that $(a, b)$ is open in $\mathbb{R}_{\geqslant 0}$.

(b) Let $a, b \in \mathbb{R}_{\geqslant 0}$ with $a < b$. Prove that $[a, b]$ is closed in $\mathbb{R}_{\geqslant 0}$.

(c) Define compact and prove that $\mathbb{R}_{\geqslant 0}$ is not compact.

(d) Define locally compact and prove that $\mathbb{R}_{\geqslant 0}$ is locally compact.

(5) (Properties of the uniform structure on $\mathbb{R}_{\geqslant 0}$)

(a) Carefully define the usual uniformity on $\mathbb{R}_{\geqslant 0}$.

(b) Define complete and prove that $\mathbb{R}_{\geqslant 0}$ is complete.

(6) (Connected and compact subsets of $\mathbb{R}_{\geqslant 0}$) Let $A \subseteq \mathbb{R}_{\geqslant 0}$.

(a) Prove that $A$ is connected if and only if $A$ is an interval.

(b) Prove that $A$ is compact if and only if $A$ is closed and bounded.

(7) (Functions on $\mathbb{R}_{\geqslant 0}$)

(a) Carefully define continuous and uniformly continuous functions.

(a) Let $n \in \mathbb{Z}_{>0}$. Prove that the function $x^n \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is continuous.

(b) Let $n \in \mathbb{Z}_{>1}$. Prove that the function $x^n \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is not uniformly continuous.

(b) Let $n \in \{0, 1\}$. Prove that the function $x^n \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is uniformly continuous.

(c) Prove that the function $e^x \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is continuous.