

Metric and Hilbert, Lecture 3D, 08 October 2015 (1)  
Univ. of Melbourne

Theorem Let  $H$  be a Hilbert space and let

$T: H \rightarrow H$  be a compact self adjoint operator.

Then there exists an orthonormal basis of eigenvectors of  $H$ .

An orthonormal sequence on  $H$  is a sequence  $(a_1, a_2, \dots)$  on  $H$  such that

$$\text{if } i, j \in \mathbb{R}_{>0} \text{ then } \langle a_i, a_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Theorem Let  $(a_1, a_2, \dots)$  be an orthonormal sequence on  $H$

$$W = \text{span}\{a_1, a_2, \dots\} \text{ and } \overline{W} = \overline{\text{span}\{a_1, a_2, \dots\}}$$

Then  $H = \overline{W} \oplus \overline{W}^\perp$  since

$$\text{if } x \in H \text{ then } x - \left( \sum_{i=1}^{\infty} \langle x, a_i \rangle a_i \right) \in \overline{W}^\perp.$$

Gram-Schmidt: Let  $H$  be a Hilbert space

Let  $(v_1, v_2, \dots)$  be sequence of linearly independent vectors in  $H$ . Then define

$$a_1 = \frac{v_1}{\|v_1\|} \quad \text{and} \quad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_n \rangle a_n - \dots - \langle v_{n+1}, a_1 \rangle a_1}{\|v_{n+1} - \langle v_{n+1}, a_n \rangle a_n - \dots - \langle v_{n+1}, a_1 \rangle a_1\|}$$

for  $n \in \mathbb{Z}_0$ . Then  $(a_1, a_2, \dots)$  is an orthonormal sequence in  $H$ .

### Eigenvectors and eigenvalues

Example  $H = \ell^2 = \{x = (x_1, x_2, \dots) \mid \|x\|_2 < \infty\}$

with  $\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$  and  $\langle \cdot, \cdot \rangle : \ell^2 \times \ell^2 \rightarrow \mathbb{R}$

given by  $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ , for  $x = (x_1, x_2, \dots)$   
 $y = (y_1, y_2, \dots)$ .

Let  $T: \ell^2 \rightarrow \ell^2$  be the linear operator given by

$$T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

(a) Show that the adjoint of  $T$  is  $T^*: \ell^2 \rightarrow \ell^2$

given by  $T^*(x_1, x_2, \dots) = (x_2, x_3, x_4, \dots)$

(since  $T e_i = e_{i+1}$  and  $\delta_{j, i+1} = \langle T e_i, e_j \rangle = \langle e_i, T^* e_j \rangle = \delta_{j-1, i}$ )

(b) The matrix of  $T$  is

$$A = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & \ddots \\ & & & & \ddots \end{pmatrix} \text{ and } A^t = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

is the matrix of  $T^*$ . Then

$$T^*T = I \text{ but } T^*T \neq I.$$

(c) If  $v = (v_1, v_2, \dots)$  is an eigenvector of  $T$  with eigenvalue  $\lambda$  then

$$\begin{aligned} \lambda v &= (\lambda v_1, \lambda v_2, \dots) \\ &= Tv = (0, v_1, v_2, \dots) \end{aligned} \quad \text{so that} \quad \begin{aligned} \lambda v_1 &= 0 \\ \lambda v_2 &= v_1 \\ \lambda v_3 &= v_2 \\ &\vdots \end{aligned}$$

So, if  $\lambda \neq 0$  then  $v = (0, 0, \dots)$  and if  $\lambda = 0$  then  $v = (0, 0, \dots)$

So  $T$  has no eigenvectors (with nonzero eigenvalue).

(d) If  $v = (v_1, v_2, \dots)$  is an eigenvector of  $T$  with eigenvalue  $\lambda$  then

$$\begin{aligned} \lambda v &= (\lambda v_1, \lambda v_2, \dots) \\ &= T^*v = (v_2, v_3, \dots) \end{aligned} \quad \text{so that} \quad \begin{aligned} v_2 &= \lambda v_1 \\ v_3 &= \lambda v_2 = \lambda^2 v_1 \\ v_4 &= \lambda v_3 = \lambda^3 v_1 \end{aligned}$$

So  $(1, \lambda, \lambda^2, \lambda^3, \dots)$  is an eigenvector of eigenvalue  $\lambda$ .

(e) Is  $T$  a compact operator?

(f) Is  $T$  a bounded operator? What is  $\|T\|$ ?

(g) Is  $T$  self adjoint?

Theorem Let  $H$  be a finite dimensional Hilbert space.  
Let  $T: H \rightarrow H$  be a linear operator.

(a)  ~~$T$~~   $T$  is bounded

(b)  $T$  is compact.

(c)  $T$  has a nonzero eigenvector.

Theorem Let  $H$  be a Hilbert space.

Let  $T: H \rightarrow H$  be a <sup>bounded</sup> compact self adjoint linear operator. Then

$T$  has a nonzero eigenvector with eigenvalue  $\|T\|$ .

Example (Eigenvalues of self adjoint linear operators are real). Let  $H$  be a Hilbert space over  $\mathbb{C}$ .

Let  $T: H \rightarrow H$  be a self adjoint operator.

Let  $v$  be a <sup>nonzero</sup> eigenvector with eigenvalue  $\lambda$ .

$$\langle Tv, v \rangle = \langle v, Tv \rangle, \quad \text{since } T \text{ is self adjoint.}$$

$$\circ \langle \lambda v, v \rangle = \langle v, \lambda v \rangle.$$

$$\circ \lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle.$$

$$\circ \lambda = \bar{\lambda}.$$