Assignment 2:

MAST30026 Metric and Hilbert Spaces
Semester II 2015
Lecturer: Arun Ram
to be turned in before 10am on 15 October 2015

(1) Let $V = \mathbb{C}^5$ with the standard Hermitian inner product. Let

$$T\colon V \to V \qquad \text{and} \qquad W\colon V \to V$$

be the linear transformations such that the matrices of T and W with respect to the standard basis of $V = \mathbb{C}^5$ are given by

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 14 & 15 \\ 16 & 0 & 2 & 0 & 20 \\ 1 & 0 & 3 & 4 & 10 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 6 & 7 & 8 & 9 \\ 3 & 7 & 10 & 11 & 12 \\ 4 & 8 & 11 & 13 & 14 \\ 5 & 9 & 12 & 14 & 15 \end{pmatrix}$$

respectively.

(a) Compute $\|T\|$ and $\|W\|$.

(b) Let T^* be the adjoint of T and let W^* be the adjoint of W. Compute the matrices of T^* and W^* with respect to the standard basis of $\mathbb{C}^5$.

(c) Show that T and W are compact operators.

(d) Find an eigenvector of W with eigenvalue $\|W\|$.

(e) Find an orthonormal basis of V which consists of eigenvectors of W.

(f) Show that T has an eigenvector.

(2) Let $(H, \langle, \rangle)$ be an infinite dimensional separable Hilbert space.

(a) Show that $H \cong \ell^2$. More precisely, show that there is an invertible linear transformation $\Phi\colon H \to \ell^2$ such that

$$\text{if } x, y \in H \quad \text{then} \quad \langle \Phi(x), \Phi(y) \rangle = \langle x, y \rangle.$$

(b) Show that ℓ^∞ is a metric space and ℓ^∞ is not separable.

(3) Let $a \in \mathbb{R}_{>0}$. Let

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \qquad \text{for } x \in \mathbb{R}_{>0}.$$

(a) Show that if $x \in \mathbb{R}_{>0}$ then $f(x) \geqslant \sqrt{a}$. Hence f defines a function $f : X \to X$ where $X = [\sqrt{a}, \infty)$.

(b) Show that f is a contraction mapping when X is given the usual metric.

(c) Fix $x_0 > \sqrt{a}$ and $x_{n+1} = f(x_n)$, for $n \in \mathbb{Z}_{\geqslant 0}$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

(4) Let $X = \mathbb{R}$ with metric given by $d(x, y) = |x - y|$.

(a) Let $A = X$. Show that A is Cauchy compact but not bounded.

(b) Let $A = X$. Show that A is Cauchy compact but not cover compact.

(c) Let $A = X$. Show that A is Cauchy compact but not ball compact.

(d) Let $A = (0, 1) \subseteq X$. Show that A is ball compact and not closed in X.

(e) Let $A = (0, 1) \subseteq X$. Show that A is ball compact and not cover compact.

(f) Let $A = (0, 1) \subseteq X$. Show that A is ball compact and not Cauchy compact.

(h) Let $A = (0, 1) \subseteq X$ and let $B = A$. Show that B is closed in A but B is not Cauchy compact.

(g) Let $Y = \mathbb{R}$ with metric given by $\rho(x, y) = \min\{|x-y|, 1\}$ and let $A = Y$. Show that A is bounded but not ball compact.