Assignment 1

MAST30026 Metric and Hilbert Spaces
Semester II 2015
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to be turned in before 10am on 10 September 2015

(1) Let (X, d) be a metric space.

- (a) Define the metric space topology $\mathcal{T}$ on X.
- (b) Define Hausdorff and show that the topological space $(X, \mathcal{T})$ is Hausdorff.
- (c) Define normal and show that the topological space $(X, \mathcal{T})$ is normal.
- (d) Define first countable and show that the topological space $(X, \mathcal{T})$ is first countable.
- (e) Give an example (with proof) of a topological space $(Y, \mathcal{U})$ which is not Hausdorff.
- (f) Give an example (with proof) of a topological space $(Y, \mathcal{U})$ which is not normal.
- (g) Give an example (with proof) of a topological space $(Y, \mathcal{U})$ which is not first countable.

(2) Let $(V, \langle, \rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{array}{ccc} V & \to & \mathbb{R}_{\geqslant 0} \\ v & \mapsto & \|v\| \end{array} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

- (a) (The Cauchy-Schwarz inequality) Show that if $x, y \in V$ then $|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|$.
- (b) (The triangle inequality) Show that if $x, y \in V$ then $\|x + y\| \leqslant \|x\| + \|y\|$.
- (c) (The Pythagorean theorem) Show that

$$\text{if } x, y \in V \text{ and } \langle x, y\rangle = 0 \qquad \text{then} \qquad \|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

- (d) (The parallelogram law) Show that

$$\text{if } x, y \in V \text{ then } \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

- (e) Show that if $(V, \| \ \|)$ is a normed vector space over $\mathbb{R}$ such that $\| \ \| \colon V \to \mathbb{R}_{\geqslant 0}$ satisfies

$$\text{if } x, y \in V \text{ then } \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle, \rangle)$ with $\langle, \rangle \colon V \times V \to \mathbb{R}$ given by

$$\langle x, y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

is a positive definite symmetric inner product space such that $\|v\|^2 = \langle v, v\rangle$. To prove that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, first establish the identity

$$\|x_1+x_2+y\| = \|x_1\|^2+\|x_2\|^2+\|x_1+y\|^2+\|x_2+y\|^2-\tfrac{1}{2}\|x_1+y-x_2\|^2-\tfrac{1}{2}\|x_2+y-x_1\|^2.$$

To prove that $\langle cx, y\rangle = \lambda cx, y\rangle$, first show that this identity holds when $c \in \mathbb{Z}$, then for $c \in \mathbb{Q}$, and finally by continuity for every $c \in \mathbb{R}$.

(f) Show that if $(V, \| \ \|)$ is a normed vector space over $\mathbb{C}$ and $\| \ \| \colon V \to \mathbb{R}_{\geqslant 0}$ satisfies

$$\text{if } x, y \in V \text{ then } \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then $(V, \langle, \rangle)$ with $\langle, \rangle \colon V \times V \to \mathbb{C}$ given by

$$\langle x, y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

is a positive definite Hermitian inner product space such that $\|v\|^2 = \langle v, v\rangle$.

(3) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $X \times Y$ have the product topology.

(a) Show that if $E \subseteq X$ then $\overline{E^c} = (E^\circ)^c$ and $(E^c)^\circ = (\overline{E})^c$.

(b) Let E be a open set in X. Show that E is a dense subset of X if and only if E^c is nowhere dense in X.

(c) Let $U_1, U_2, \ldots$ be open dense subsets of X. Show that $\displaystyle\bigcup_{i \in \mathbb{Z}_{>0}} U_i$ is dense in X if and only if $\displaystyle\bigcap_{i \in \mathbb{Z}_{>0}} (U_i)^c$ has empty interior.

(d) Show that an open set in $X \times Y$ cannot be expected to be of the form $A \times B$ with A open in X and B open in Y.

(e) Show that if $A \subseteq X$ and $B \subseteq Y$ then

$$\overline{A} \times \overline{B} = \overline{A \times B} \qquad \text{and} \qquad A^\circ \times B^\circ = (A \times B)^\circ.$$

(4) Let $p \in \mathbb{R}_{\geqslant 1}$ and define

$$\ell^p = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_p < \infty\}, \quad \text{where} \qquad \|\vec{x}\|_p = \left(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p\right)^{1/p}$$

for a sequence $\vec{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$.

(a) Show that if $p \leqslant q$ then $\ell^p \subseteq \ell^q$.

(b) Show that if $p \neq q$ then $\ell^p \neq \ell^q$.

(5) Carefully define $B(V,W)$ and prove that if W is complete then $B(V,W)$ is complete.

(6) (sequences of functions) Let (X,d) and (C,ρ) be metric spaces. Let

$$F = \{\text{functions } f\colon X \to C\} \qquad \text{and define} \quad d_\infty\colon F \times F \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f,g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(Warning d_∞ is not quite a metric since its target is not $\mathbb{R}_{\geqslant 0}$.) Let

$$(f_1, f_2, \ldots) \text{ be a sequence in } F \quad \text{and let} \quad f\colon X \to C$$

be a function.

The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

if $x \in X$ and $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geqslant N}$ then $d(f_n(x), f(x)) < \epsilon$.

The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

if $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that
if $x \in X$ and $n \in \mathbb{Z}_{\geqslant N}$ then $\rho(f_n(x), f(x)) < \epsilon$.

(a) Show that $(f_1, f_2, \ldots)$ converges pointwise to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} \rho(f_n(x), f(x)) = 0.$$

(b) Show that $(f_1, f_2, \ldots)$ converges uniformly to f if and only if $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

(7) For a topological space X and a sequence $\vec{x} = (x_1, x_2, \ldots)$ in X write

$$y = \lim_{n\to\infty} x_n, \qquad \begin{array}{l}\text{if } y \text{ is a limit point of } \vec{x}\colon \mathbb{Z}_{>0} \to X \\ \text{with respect to the tail filter on } \mathbb{Z}_{>0}.\end{array}$$

(a) Let X and Y be topological spaces. Define what it means for a function $f\colon X \to Y$ to be continuous.

(b) Let X and Y be uniform spaces. Define what it means for a function $f\colon X \to Y$ to be uniformly continuous.

(c) Let X and Y be uniform spaces. Show that if $f\colon X \to Y$ uniformly continuous then $f\colon X \to Y$ is continuous.

(d) Let (X,d) and (Y,ρ) be metric spaces and let $f\colon X \to Y$ be a function. Show that $f\colon X \to Y$ is continuous if and only if f satisfies

if $\epsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that
if $y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \epsilon$.

(e) Let (X, d) and (Y, ρ) be metric spaces and let $f: X \to Y$ be a function. Show that $f: X \to Y$ is uniformly continuous if and only if f satisfies

$$\begin{aligned}&\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that}\\ &\qquad \text{if } x, y \in X \text{ and } d(x,y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon.\end{aligned}$$

(f) Let (X, d) and (Y, ρ) be metric spaces and let $f: X \to Y$ be a function. Show that f is continuous if and only if f satisfies

$$\text{if } (x_1, x_2, \ldots) \text{ is a sequence in } X \text{ and } \lim_{n\to\infty} x_n \text{ exists} \quad \text{then} \quad f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n).$$

(8) Let C be the Cantor set and let $Q = \{x \in \mathbb{Q} \mid 0 \leqslant x \leqslant 1\}$. Let C and Q have the subspace topology of the interval $X = [0,1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$ in $\mathbb{R}$, where $\mathbb{R}$ has the standard topology.

(a) Show that C is closed in X and not open in X, and Q is not closed in X and Q is not open in X.

(b) Show that C is nowhere dense in X and Q is dense in X.

(c) Show that C^c is dense in X and Q^c is dense in X.

(d) Show that C is compact and Q is not compact.

(e) Show that C and Q are both totally disconnected (i.e. every connected component is a set with a single point).

(e) Let μ be a function which assigns values to certain subsets of X which satisfies

$$\mu([a,b]) = b - a, \quad \text{if } a, b \in \mathbb{R} \text{ and } 0 \leqslant a < b \leqslant 1,$$

and

$$\mu\Big(\bigcup_{i \in \mathbb{Z}_{>0}} A_i\Big) = \sum_{i \in \mathbb{Z}_{>0}} \mu(A_i) \quad \text{if } A_1, A_2, \ldots \text{ are disjoint subsets of } X\ .$$

Show that

$$\mu(C) = 0, \quad \mu(C^c) = 1, \quad \mu(Q) = 0, \quad \text{and} \quad \mu(Q^c) = 1.$$

(f) Show that $\mathrm{Card}(C) = \mathrm{Card}(\mathbb{R})$, $\mathrm{Card}(C^c) = \mathrm{Card}(\mathbb{R})$, $\mathrm{Card}(Q) \neq \mathrm{Card}(\mathbb{R})$ and $\mathrm{Card}(Q^c) = \mathrm{Card}(\mathbb{R})$.