Tutorial Sheet 2

MAST30026 Metric and Hilbert Spaces
Semester II 2015
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(1) (positive definite inner product spaces are normed vector spaces) Let $(V, \langle , \rangle)$ be a positive definite inner product space. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\geqslant 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{given by} \qquad \|v\|^2 = \langle v, v\rangle.$$

Show that $(V, \|\ \|)$ is a normed vector space.

(2) (normed vector spaces are metric spaces) Let $(V, \|\ \|)$ be a normed vector space. The *norm metric* on V is the function

$$d\colon V \times V \to \mathbb{R}_{\geqslant 0} \qquad \text{given by} \qquad d(x, y) = \|x - y\|.$$

Show that (V, d) is a metric space.

(3) (uniformity of a pseudometric) [Bou, Top. Ch. IX §1 no. 2] Let X be a set. A *pseudometric on* X is a function $f\colon X \times X \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that

(a) If $x \in X$ then $d(x, x) = 0$,
(b) If $x, y \in X$ then $d(x, y) = d(y, x)$,
(c) If $x, y, z \in X$ then $d(x, y) \leqslant d(x, z) + d(z, y)$.

Show that the sets

$$B_\epsilon = \{(x, y) \in X \times X \mid d(x, y) \leqslant \epsilon\}, \qquad \text{for } \epsilon \in \mathbb{R}_{>0},$$

generate a uniformity $\mathcal{X}_d$ on X.

(4) (The uniform space topology is a topology) Let $(X, \mathcal{X})$ be a uniform space. Let

$$B_V(x) = \{y \in X \mid (x, y) \in V\} \quad \text{for } V \in \mathcal{X} \text{ and } x \in X, \quad \text{and let}$$
$$\mathcal{N}(x) = \{B_V(x) \mid V \in \mathcal{X}\} \quad \text{for } x \in X.$$

Show that $\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then } U \in \mathcal{N}(x)\}$ is a topology on X.

(5) (The metric space topology is a topology) Let (X, d) be a metric space. Let

$$B_\epsilon(x) = \{y \in X \mid d(y,x) < \epsilon\} \quad \text{for } \epsilon \in \mathbb{R}_{>0} \text{ and } x \in X.$$

Let $\mathcal{B} = \{B_\epsilon(x) \mid \epsilon \in \mathbb{R}_{>0}, x \in X\}$.

(a) Show that $\mathcal{T} = \{\text{unions of sets in } \mathcal{B}\}$ is a topology on X.

(b) Show that if $\mathcal{U}$ is a topology on X and $\mathcal{U} \supseteq \mathcal{B}$ then $\mathcal{U} = \mathcal{T}$.

(6) (consistency of metric space topology, uniform space topology and metric space uniformity) Let (X, d) be a metric space and let $\mathcal{X}$ be the metric space uniformity on X. Show that the uniform space topology of $(X, \mathcal{X})$ is the same as the metric space topology on (X, d).

(7) Give an example of a topological space that is not a uniform space.

(8) Give an example of a uniform space that is not a metric space.

(9) Give an example of a metric space that is not a normed vector space.

(10) Give an example of a normed vector space that is not a positive definite inner product space.

(11) (Lipschitz equivalence implies topological equivalence) Let X be a set and let

$$d_1 \colon X \times X \to \mathbb{R}_{\geqslant 0} \qquad \text{and} \qquad d_2 \colon X \times X \to \mathbb{R}_{\geqslant 0} \qquad \text{be metrics on } X.$$

The metrics d_1 and d_2 are *topologically equivalent* if

the metric space topology on (X, d_1) and on (X, d_2) are the same.

The metrics d_1 and d_2 are *Lipschitz equivalent* if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$\text{if } x, y \in X \quad \text{then} \quad c_1 d_2(x,y) \leqslant d_1(x,y) \leqslant c_2 d_1(x,y).$$

Show that if d_1 and d_2 are Lipschitz equivalent then d_1 and d_2 are topologically equivalent.

(12) (every metric space is topologically equivalent to a bounded metric space) A metric space (X, d) is *bounded* if it satisfies

$$\text{there exists } M \in \mathbb{R}_{\geqslant 0} \text{ such that} \quad \text{if } x_1, x_2 \in X \text{ then } d(x_1, x_2) < M.$$

Let (X, d) be a metric space and define $b \colon X \times X \to \mathbb{R}_{\geqslant 0}$ by

$$b(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

(a) Show that $b\colon X \times X \to \mathbb{R}_{\geqslant 0}$ is a metric on X.

(b) Show that the metric space topology of (X, b) and the metric space topology on (X, d) are the same.

(c) Show that (X, b) is a bounded metric space.

(13) (boundedness is not a topological property) A metric space (X, d) is *bounded* if it satisfies

$$\text{there exists } M \in \mathbb{R}_{\geqslant 0} \text{ such that} \quad \text{if } x_1, x_2 \in X \text{ then } d(x_1, x_2) < M.$$

Let $X = \mathbb{R}$ and let $d\colon X \times X \to \mathbb{R}_{\geqslant 0}$ and $b\colon X \times X \to \mathbb{R}_{\geqslant 0}$ be the metrics on $\mathbb{R}$ given by

$$d(x, y) = |x - y| \qquad \text{and} \qquad b(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Show that $(\mathbb{R}, d)$ and $(\mathbb{R}, b)$ have the same topology, that $(\mathbb{R}, d)$ is unbounded, and $(\mathbb{R}, b)$ is bounded.

(14) (composition of continuous functions is continuous) Continuous functions are for comparing topological spaces. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. A *continuous function from X to Y* is a function $f\colon X \to Y$ such that

$$\text{if } V \text{ is an open set of } Y \quad \text{then} \quad f^{-1}(V) \text{ is an open set of } X,$$

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions. Show that $g \circ f$ is continuous.

(15) (composition of uniformly continuous functions is uniformly continuous) Uniformly continuous functions are for comparing uniform spaces. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be uniform spaces. A *uniformly continuous function from X to Y* is a function $f\colon X \to Y$ such that

$$\text{if } W \in \mathcal{Y} \text{ then there exists } V \in \mathcal{X} \text{ such that} \qquad \text{if } (x, y) \in V \text{ then } (f(x), f(y)) \in W.$$

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be uniformly continuous functions. Show that $g \circ f$ is uniformly continuous.

(16) (continuous is the same as continuous at each point) Let X and Y be topological spaces and let $a \in X$. A function $f\colon X \to Y$ is *continuous at a* if f satisfies the condition

$$\text{if } V \text{ is a neighborhood of } f(a) \text{ in } Y \quad \text{then} \quad f^{-1}(V) \text{ is a neighborhood of } a \text{ in } X.$$

Let X and Y be topological spaces and let $f\colon X \to Y$ be a function. Show that f is continuous if and only if

$$f \text{ satisfies:} \quad \text{if } a \in X \quad \text{then} \quad f \text{ is continuous at } a.$$

(17) (continuous images of connected sets are connected and continuous images of compact sets are compact) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The set E is *connected* if there do not exist open sets A and B in X ($A, B \in \mathcal{T}$) with

$$A \cap E \neq \emptyset \quad \text{and} \quad B \cap E \neq \emptyset \quad \text{and} \quad A \cup B \supseteq E \quad \text{and} \quad (A \cap B) \cap E = \emptyset.$$

The set E is *compact* if E satisfies

$$\text{if } \mathcal{S} \subseteq \mathcal{T} \text{ and } E \subseteq \left(\bigcup_{U \in \mathcal{S}} U \right) \quad \text{then there exists}$$
$$\ell \in \mathbb{Z}_{>0} \text{ and } U_1, U_2, \ldots, U_\ell \in \mathcal{S} \text{ such that } E \subseteq U_1 \cup U_2 \cup \cdots \cup U_\ell.$$

Let $f : X \to Y$ be a continuous function and let $E \subseteq X$. Show that

(a) If E is connected then $f(E)$ is connected,

(b) If E is compact then $f(E)$ is compact.