Tutorial Sheet 1

MAST30026 Metric and Hilbert Spaces
Semester II 2015
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(1) Define the necessary terms and prove the following statement:

A function $f: X \to Y$ is invertible if and only if f is a bijection.

(2) Let S be a set. Define the necessary terms and carefully state and prove a result that establishes that the data of an equivalence relation on S and a partition of S are equivalent data.

(3) Define the necessary terms and establish (with proof) the following:

- (a) Show that if S is a poset and E is a subset of S and a greatest lower bound of E exists then it is unique.
- (b) True or false: If S is a poset and E is a subset of S then a greatest lower bound of E exists.
- (c) True or false: If S is a poset and E is a subset of S and a minimal element of E exists then it is unique.
- (d) True or false: If S is a poset and E is a subset of S then a minimal element of E exists.
- (e) True or false: If S is a poset and E is a subset of S then a largest element of E exists.
- (f) True or false: If S is a poset and E is a subset of S and a largest element of E exists then it is unique.

(4) Define the necessary terms and establish (with proof) the following:

- (a) Show that if S is a right filtered poset and a is a maximal element of S then a is the largest element of S.
- (b) Show that every well ordered set is totally ordered.
- (c) Show that there exist totally ordered sets that are not well ordered.
- (d) Show that if S is a lattice then the intersection of two intervals is an interval.
- (e) Give an example of a poset X such that the collection $\mathcal{T} = \{\text{unions of open intervals}\}$ is not a topology.

(f) Show that if X is the poset $X = \mathbb{R}$ then the collection $\mathcal{T} = \{\text{unions of open intervals}\}$ is the standard topology on $\mathbb{R}$.

(5) Let $\mathbb{F}$ be a field. Define the necessary terms (including field) and establish (with proof) the following:

(a) If $a \in \mathbb{F}$ then $a \cdot 0 = 0$.
(b) If $a \in \mathbb{F}$ then $-(-a) = a$.
(c) If $a \in \mathbb{F}$ and $a \neq 0$ then $(a^{-1})^{-1} = a$.
(d) If $a \in \mathbb{F}$ then $a(-1) = -a$.
(e) If $a, b \in \mathbb{F}$ then $(-a)b = -ab$.
(f) If $a, b \in \mathbb{F}$ then $(-a)(-b) = ab$.

(6) Let $\mathbb{F}$ be an ordered field. Define the necessary terms (including field) and establish (with proof) the following:

(a) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.
(b) If $a \in \mathbb{F}$ and $a \neq 0$ then $a^2 > 0$.
(c) $1 \geqslant 0$.
(d) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.
(e) If $a, b \in \mathbb{F}$ and $a \geqslant 0$ and $b \geqslant 0$ then $a + b \geqslant 0$.
(f) If $a, b \in \mathbb{F}$ and $0 < a < b$ then $b^{-1} < a^{-1}$.

(7) Define $\mathbb{R}_{\geqslant 0}$ and establish (with proof) the following

(a) If $x, y, z \in \mathbb{R}_{\geqslant 0}$ then $(x + y) + z = x + (y + z)$.
(b) If $x \in \mathbb{R}_{\geqslant 0}$ then $0 + x = x$ and $x + 0 = x$.
(c) If $x, y \in \mathbb{R}_{\geqslant 0}$ then $x + y = y + x$.
(d) If $x, y, z \in \mathbb{R}_{\geqslant 0}$ then $(xy)z = x(yz)$.
(e) If $x \in \mathbb{R}_{\geqslant 0}$ then $0 \cdot x = x$ and $x \cdot 1 = x$.
(f) If $x \in \mathbb{R}_{\geqslant 0}$ and $x \neq 0$ then there exists $x^{-1} \in \mathbb{R}_{\geqslant 0}$ such that $x \cdot x^{-1} = 1$ and $x^{-1} \cdot x = 1$.
(g) If $x, y \in \mathbb{R}_{\geqslant 0}$ then $xy = yx$.
(h) If $x, y, z \in \mathbb{R}_{\geqslant 0}$ then $x(y + z) = xy + xz$.
(i) If $x, y, z \in \mathbb{R}_{\geqslant 0}$ and $x \leqslant y$ then $x + z \leqslant y + z$.
(j) If $x, y \in \mathbb{R}_{\geqslant 0}$ then $xy \in \mathbb{R}_{\geqslant 0}$.

(8) Let $n \in \mathbb{Z}_{>0}$. Define the necessary terms and establish the following:

(a) The function $x^n \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is continuous.
(b) The function $x^n \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is bijective.

(c) The function $x^n\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ satisfies

$$\text{if } x, y \in \mathbb{R}_{\geqslant 0} \text{ and } x < y \quad \text{then} \quad x^n < y^n.$$

(d) The inverse function $x^{\frac{1}{n}}\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ to the function $x^n\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is continuous.

(9) Show that the functions

$$\begin{matrix} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & -x \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (x,y) & \mapsto & xy \end{matrix} \qquad \text{are continuous.}$$

Determine (with proof) which of these functions are uniformly continuous.

(10) Let $x, y \in \mathbb{R}^n$. Define the necessary terms (including $|x|$ and $\langle x, y\rangle$) and establish the following:

(a) (Lagrange's identity) $|x|^2 \cdot |y|^2 - \langle x, y\rangle^2 = \frac{1}{2}\sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2$.

(b) (Cauchy-Schwarz inequality) $\langle x, y\rangle \leqslant |x| \cdot |y|$.

(c) (triangle inequality) $|x + y| \leqslant |x| + |y|$.

(11) Let $(V, \langle,\rangle)$ be a positive definite inner product space. Define the necessary terms (including positive definite inner product space and $\|x\|$) and establish the following:

(a) (Cauchy-Schwarz inequality) If $x, y \in V$ then $|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|$.

(b) (triangle inequality) If $x, y \in V$ then $\|x + y\| \leqslant \|x\| + \|y\|$.

(12) Let $q \in \mathbb{R}_{\geqslant 1}$ and let $p \in \mathbb{R}_{>1} \cup \{\infty\}$ be given by $\frac{1}{p} + \frac{1}{q} = 1$. Define the necessary terms and establish the following:

(a) (Young's inequality) If $a, b \in \mathbb{R}_{>0}$ then $a^{\frac{1}{p}}\, b^{\frac{1}{q}} \leqslant \frac{1}{p}\, a + \frac{1}{q}\, b$.

(b) (Hölder inequality for $\mathbb{R}^n$) If $x, y \in \mathbb{R}^n$ then $|\langle x, y\rangle| \leqslant \|x\|_p \|y\|_q$.

(c) (Minkowski inequality for $\mathbb{R}^n$) If $x, y \in \mathbb{R}^n$ then $\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$.

(d) (Hölder inequality) If $x \in \ell^p$ and $y \in \ell^q$ then $|\langle x, y\rangle| \leqslant \|x\|_p \|y\|_q$.

(e) (Minkowski inequality) If $x \in \ell^p$ and $y \in \ell^q$ then $\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$.