"Around loop groups, Langlands and mathematical physics"

Lecture 9: <u>W-algebras</u>, Univ. of Melbourne 22 April 2015

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<u>Slodowy slices S_f</u>

$\mathring{\mathfrak{g}}$ is a finite dim'l simple Lie algebra over $\mathbb{C}$.

$$(\,|\,): \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C} \quad \text{and} \quad \nu: \begin{array}{ccc} \mathring{\mathfrak{g}} & \xrightarrow{\sim} & \mathring{\mathfrak{g}}^* \\ x & \longmapsto & (x|\cdot) \end{array}$$

where $(\,|\,)$ is a nondeg. ad-inv. symm. bilinear form.

$f \in \mathring{\mathfrak{g}}$ is nilpotent, $\{e, f, h\}$ an $\mathfrak{sl}_2$-triple.

The <u>Slodowy slice</u> is

$$S_f = \nu(f + Z_{\mathring{\mathfrak{g}}}(e)) \quad \text{containing} \quad \chi = \nu(f) \in \mathring{\mathfrak{g}}^*.$$

W-algebras and BRST reduction

Let $M \in \mathbb{C}[\mathring{\mathfrak{g}}^*]$-Pmod and

let $H_f^\bullet(M)$ be the cohomology of $(\bar{C}(M), \mathrm{ad}\,\bar{d})$

<u>Arakawa Theorem 2.2</u> (see Kostant-Sternberg 87, DeSole-Kac 06)

As Poisson algebras,

$$H_f^0(\mathbb{C}[\mathring{\mathfrak{g}}^*]) \simeq \mathbb{C}[S_f] \qquad \begin{pmatrix}\text{classical BRST} \\ \text{reduction}\end{pmatrix}$$

and $H_f^i(\mathbb{C}[\mathring{\mathfrak{g}}^*]) = 0$ for $i \neq 0$.

Let $M \in \mathcal{HC}$ and

let $H_f^\bullet(M)$ be the cohomology of $(C(M), \mathrm{ad}\, d)$

The <u>finite W-algebra of $(\mathring{\mathfrak{g}}, f)$</u> is

the associative algebra

$$U(\mathring{\mathfrak{g}}, f) = H_f^0(U\mathring{\mathfrak{g}}) \qquad \begin{pmatrix}\text{BRST} \\ \text{reduction}\end{pmatrix}$$

Let M be a $V^k(\mathring{\mathfrak{g}})$-module and

let $H_f^\bullet(M)$ be the cohomology of $(C^{ch}(M), Q_{(0)})$.

The <u>W-algebra of $(\mathring{\mathfrak{g}}, f)$ at level k</u> is

the vertex algebra

$$\mathcal{W}^k(\mathring{\mathfrak{g}}, f) = H_f^0(V^k(\mathring{\mathfrak{g}})) \qquad \begin{pmatrix}\text{quantised} \\ \text{Drinfeld-Sokolov} \\ \text{reduction}\end{pmatrix}$$

$U(\mathring{\mathfrak{g}},f)$ as an endomorphism algebra

Let $\{e,f,h\}$ be the sl_2-triple of f.

$$\mathring{\mathfrak{g}} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathring{\mathfrak{g}}_j, \text{ where } \mathring{\mathfrak{g}}_j = \{x\in\mathring{\mathfrak{g}} \mid [h,x]=2jx\}$$

Let

$$\mathring{\mathfrak{g}}_{>0} = \bigoplus_{j>0} \mathring{\mathfrak{g}}_j = \mathring{\mathfrak{g}}_{\frac{1}{2}} \oplus \mathring{\mathfrak{g}}_{\geq 1}.$$

Define a symplectic form

$$\mathring{\mathfrak{g}}_{\frac{1}{2}} \times \mathring{\mathfrak{g}}_{\frac{1}{2}} \longrightarrow \mathbb{C} \quad \text{by} \quad \langle x,y\rangle = \chi([x,y]).$$

Choose a Lagrangian subspace

$$\ell \subseteq \mathring{\mathfrak{g}}_{\frac{1}{2}} \text{ and let } \mathfrak{m} = \ell \oplus \mathring{\mathfrak{g}}_{\geq 1}.$$

Then $\mathfrak{m}$ is a nilpotent subalgebra of $\mathring{\mathfrak{g}}_{>0}$ and

$$\chi : \mathfrak{m} \to \mathbb{C} \quad \text{is a character.}$$

<u>Thm</u>

$$U(\mathring{\mathfrak{g}},f) \cong \mathrm{End}_{U\mathring{\mathfrak{g}}}(Y)^{op}, \text{ where } Y = U\mathring{\mathfrak{g}} \otimes_{U\mathfrak{m}} \mathbb{C}v_\chi$$

with $yv_\chi = \chi(y)v_\chi$ for $y\in\mathfrak{m}$.

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Affine W-algebras to finite W-algebras and Slodowy slices

Let V be a vertex algebra.

<u>Zhu's C_2-algebra of V</u> is the Poisson algebra

$$R_V = \frac{V}{C_2(V)} \quad \text{with } \bar{a}\cdot\bar{b} = \overline{a_{(-1)}b} \text{ and } \{\bar{a},\bar{b}\} = \overline{a_{(0)}b}$$

for $a,b\in V$, where

$$C_2(V) = \mathbb{C}\text{-span}\{a_{(-2)}b \mid a,b\in V\}$$

The (L_0-twisted) <u>Zhu's algebra of V</u> is the assoc. alg.

$$A(V) = \frac{V}{O(V)} \quad \text{with } a*b = \sum_{i\in\mathbb{Z}_{\geq 0}} \binom{\mathrm{wt}(a)}{i} a_{(i-1)}b$$

for $a,b\in V$, where

$$O(V) = \mathbb{C}\text{-span}\left\{ a\circ b = \sum_{i\in\mathbb{Z}_{\geq 0}} \binom{\mathrm{wt}(a)}{i} a_{(i-2)}b \;\middle|\; \begin{matrix}\text{for homogeneous}\\ a,b\in V\end{matrix} \right\}$$

<u>Arakawa before Theorem 7.1</u>

$$\bar{\eta}_{V^k(\mathring{g})} : R_{\mathcal{W}^k(\mathring{g},f)} \xrightarrow{\sim} \mathbb{C}[S_f] \qquad \left(\begin{matrix}\text{Poisson}\\ \text{algebras}\end{matrix}\right)$$

<u>Arakawa equation (31)</u>

$$\eta_{V^k(\mathring{g})} : A(\mathcal{W}^k(\mathring{g},f)) \xrightarrow{\sim} U(\mathring{g},f) \qquad \left(\begin{matrix}\text{associative}\\ \text{algebras}\end{matrix}\right)$$

Vertex algebras

A vertex algebra is a vectorspace V with a vacuum $\mathbb{1} \in V$, $T \in End(V)$, and a linear map $Y(\cdot, z): V \to End(V)[z, z^{-1}]$

$$Y(a,z) = a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \qquad (a_{(n)} \in End(V))$$

such that

(a) $\mathbb{1}(z) = id_V$,

(b) If $a \in V$ then $a_{(-1)} \mathbb{1} = a$,

(c) If $a, b \in V$ then there exists $N \in \mathbb{Z}_{\geq 0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $a_{(n)} b = 0$,

(d) If $a \in V$ then $(Ta)(z) = [T, a(z)] = \frac{d}{dz} a(z)$,

(e) If $a, b \in V$ then there exists $N \in \mathbb{Z}_{\geq 0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $(z-w)^n [a(z), b(w)] = 0$ in $End(V)$.

A conformal vertex algebra is a vertex algebra V with a conformal vector $\omega \in V$, such that

if $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ then there exists

a central charge $c_V \in \mathbb{C}$ with

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n} c_V,$$

$L_{-1} = T$ and L_0 diagonalizable on V.

The universal affine vertex algebra $V^k(\mathring{\mathfrak{g}})$

The affine Lie algebra is

$$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K = \mathring{\mathfrak{g}}[t,t^{-1}] \oplus \mathbb{C}K.$$

The <u>universal affine vertex algebra of $\mathring{\mathfrak{g}}$ at level k</u> is

$$V^k(\mathring{\mathfrak{g}}) = U\mathfrak{g} \otimes_{U(\mathring{\mathfrak{g}}[t] \oplus \mathbb{C}K)} \mathbb{C}v_k$$

where

$$Kv_k = kv_k \quad \text{and} \quad xt^m v_k = 0 \quad \text{for } x \in \mathring{\mathfrak{g}},\ m \in \mathbb{Z}_{\geq 0}.$$

with vacuum $\mathbb{1} = 1 \otimes v_k$,

conformal vector $\omega_{\mathring{\mathfrak{g}}} = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathring{\mathfrak{g}}} (x_i t^{-1})(x^i t^{-1})\mathbb{1}$, and

$$Y(xt^{-1}\mathbb{1}, z) = \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1}, \quad \text{for } x \in \mathring{\mathfrak{g}}.$$

Arakawa equations (25) and (31)

$$A(V^k(\mathring{\mathfrak{g}})) = U\mathring{\mathfrak{g}},$$

$$A(W^k(\mathring{\mathfrak{g}}, f)) = U(\mathring{\mathfrak{g}}, f)$$

Poisson (classical) BRST reduction

Category: $\overline{\mathcal{HC}} = \left\{ \begin{array}{l} \mathbb{C}[\mathring{\mathfrak{g}}^*] \text{ Poisson modules } M \text{ on which} \\ \text{the } \mathring{\mathfrak{g}}\text{-action is locally finite} \end{array} \right\}$

Poisson Weyl algebra:

$$\overline{\mathcal{D}} = \mathbb{C}[\chi + \nu^{-1}(\mathring{\mathfrak{g}}_{-\frac{1}{2}})] = \frac{\mathbb{C}[\mathring{\mathfrak{g}}^*_{>0}]}{\langle x - \chi(x) \mid x \in \mathring{\mathfrak{g}}_{\geq 1} \rangle}$$

Poisson Clifford algebra:

$$\overline{\mathcal{Cl}} = \mathbb{C}[T^*\Pi\mathring{\mathfrak{g}}^*_{>0}] = \Lambda^\circ(\mathring{\mathfrak{g}}^*_{>0} \oplus \mathring{\mathfrak{g}}_{>0}),$$

where $\Pi\mathring{\mathfrak{g}}^*_{>0} = \mathring{\mathfrak{g}}^*_{>0}$ as a purely odd vector space.

Poisson BRST complex:

$$\overline{C}(M) = M \otimes \overline{\mathcal{D}} \otimes \overline{\mathcal{Cl}} = \bigoplus_{p \in \mathbb{Z}} \overline{C}^p(M)$$

with

$$\overline{C}^p(M) = \bigoplus_{i-j=p} M \otimes \overline{\mathcal{D}} \otimes \Lambda^i \mathring{\mathfrak{g}}^*_{>0} \otimes \Lambda^j \mathring{\mathfrak{g}}_{>0}.$$

Poisson BRST differential:

$$\overline{d} = \sum_{i=1}^r (x_i \otimes 1 + 1 \otimes \Phi_i) \otimes x_i^* - 1 \otimes 1 \otimes \frac{1}{2} \sum_{k=1}^r c_{ij}^k x_i^* x_j^* x_k$$

where $\{x_1, \dots, x_r\}$ is a homogeneous basis of $\mathring{\mathfrak{g}}_{>0}$,

$\{x_1^*, \dots, x_r^*\}$ is the dual basis in $\mathring{\mathfrak{g}}^*_{>0}$

$\{\Phi_1, \dots, \Phi_r\}$ are the images of $\{x_1, \dots, x_r\}$ in $\overline{\mathcal{D}}$

and $$[x_i, x_j] = \sum_{k=1}^r c_{ij}^k x_k$$

BRST reduction

Category: $\mathcal{HC} = \left\{ \begin{array}{l} U\mathring{\mathfrak{g}}\text{-bimodules } M \text{ on which the} \\ \text{adjoint } \mathring{\mathfrak{g}}\text{-action is locally finite} \end{array} \right\}$

Weyl algebra:
$$\mathcal{D} = \frac{U\mathring{\mathfrak{g}}_{>0}}{\sum_{x \in \mathring{\mathfrak{g}}_{\geq 1}} U\mathring{\mathfrak{g}}_{>0}(x - \chi(x))}$$

Clifford algebra: Cl is the Clifford algebra of $\mathring{\mathfrak{g}}_{>0}^* \oplus \mathring{\mathfrak{g}}_{>0}$ with bilinear form $(x+f, x'+f') = f(x') + f'(x)$ so that the multiplication map

$$\Lambda^\bullet(\mathring{\mathfrak{g}}_{>0}^*) \otimes \Lambda^\bullet(\mathring{\mathfrak{g}}_{>0}) \to Cl \text{ is a vector sp. isom.}$$

BRST complex:
$$C(M) = M \otimes \mathcal{D} \otimes Cl = \bigoplus_{p \in \mathbb{Z}} C^p(M)$$

with
$$C(M) = \bigoplus_{i-j=p} M \otimes \mathcal{D} \otimes \Lambda^i \mathring{\mathfrak{g}}_{>0}^* \otimes \Lambda^j \mathring{\mathfrak{g}}_{>0}$$

BRST differential:
$$d = \sum_{i=1}^{r} (x_i \otimes 1 + 1 \otimes \phi_i) \otimes x_i^* - 1 \otimes 1 \otimes \frac{1}{2} \sum_{i,j,k=1}^{r} c_{ij}^k x_i^* x_j^* x_k$$

where $\{\phi_1, \dots, \phi_r\}$ are the images of $\{x_1, \dots, x_r\}$ in $\mathcal{D}$.

For $M \in \mathcal{HC}$ let

$H_f^\bullet(M)$ be the cohomology of the complex $(C(M), \operatorname{ad} d)$

Quantized Drinfeld-Sokolov reduction

Category: $\{V^k(\mathfrak{g})\text{-modules } M\} = \left\{\begin{matrix}\text{smooth } \mathfrak{g}\text{-modules} \\ \text{of level } k\end{matrix}\right\}$

Vertex Weyl algebra: $\{x_1, \dots, x_s\}$ a basis of $\mathring{\mathfrak{g}}_{\frac{1}{2}}$

D^{ch} is the $\beta\gamma$ system of rank s,

freely generated by $\phi_1(z), \dots, \phi_s(z)$

satisfying OPE (operator product expansions)

$$\phi_i(z)\phi_j(w) \sim \frac{\chi([x_i; x_j])}{z-w}$$

Vertex Clifford algebra: $\{x_1, \dots; x_s, x_{s+1}, \dots, x_r\}$ a basis of $\mathring{\mathfrak{g}}_{>0}$.

$\Lambda^{\frac{\infty}{2}+\bullet}$ is a vertex superalgebra generated by

$\psi_1(z), \dots, \psi_r(z)$ and $\psi_1^*(z), \dots, \psi_r^*(z)$

satisfying OPE

$$\psi_i(z)\psi_j^*(w) \sim \frac{\delta_{ij}}{z-w} \quad \text{and} \quad \psi_i(z)\psi_j(w) \sim \psi_i^*(z)\psi_j^*(w) \sim 0.$$

BRST complex: $C^{ch}(M) = M \otimes D^{ch} \otimes \Lambda^{\frac{\infty}{2}+\bullet}$

BRST Drinfeld-Sokolov differential:

$$Q(z) = \sum_{n\in\mathbb{Z}} Q_{(n)} z^{-n-1}$$

$$= \sum_{i=1}^{r} (x_i(z) + \phi_i(z))\psi_i^*(z) - \frac{1}{2}\sum_{i,j,k=1}^{r} c_{ij}^k \psi_i^*(z)\psi_j^*(z)\psi_k(w)$$

where $x_i(z) = Y(x_i t^{-1}\mathbb{1}, z) = \sum_{n\in\mathbb{Z}} (x_i t^n) z^{-n-1}$.