"Around loop groups Langlands and mathematical physics"
Lecture 8, Fusion algebras, 15 April 2015, University of Melbourne (1)

Symmetric functions $\mathcal{S} = \mathbb{C}[\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}]^{W_0}$ Arun Ram

$\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}$ has $\mathbb{Z}$-basis $\omega_1, \ldots, \omega_n$ (the fundamental weights).

The group algebra of $\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}$ is

$$\mathbb{C}[\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}] = \mathrm{span}\{e^{\gamma} \mid \gamma \in \mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}\} \quad \text{with } e^{\gamma}e^{\tau} = e^{\gamma+\tau}.$$

If $X_i = e^{\omega_i}$ then $\mathbb{C}[\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}] = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

The Weyl group W_0 acts on $\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}$ (a finite group generated by reflections $s_1, \ldots, s_n$). Then

$$W_0 \text{ acts on } \mathbb{C}[\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}] \text{ by } \quad we^{\gamma} = e^{w\gamma}.$$

and

$$\mathcal{S} = \mathbb{C}[\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}]^{W_0} = \{f \in \mathbb{C}[\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}] \mid wf = f \text{ for } w \in W_0\}$$

The Weyl characters, or Schur functions, are

$$s_\lambda = \frac{\sum_{w \in W_0} \det(w) e^{w(\lambda+\rho)}}{\sum_{w \in W_0} \det(w) e^{w\rho}} \qquad \text{where } \rho = \omega_1 + \cdots + \omega_n$$

Then $\{s_\lambda \mid \lambda \in P^+\}$, where $P^+ = \mathbb{Z}_{\geq 0}\omega_1 + \cdots + \mathbb{Z}_{\geq 0}\omega_n = \mathring{\mathfrak{h}}_{\mathbb{Z}_{\geq 0}}^{*}$, form a basis of $\mathbb{C}[\mathring{\mathfrak{h}}_{\mathbb{Z}}^{*}]^{W_0}$ and $c_{\lambda\mu}^{\nu}$ defined by

$$s_\lambda s_\mu = \sum_{\nu \in P^+} c_{\lambda\mu}^{\nu} s_\nu$$

are the structure constants of $\mathcal{S}$ in the basis $\{s_\lambda \mid \lambda \in P^+\}$, or the Littlewood-Richardson coefficients, or the Clebsch-Gordan coefficients, or the tensor product multiplicities.

<u>The fusion rings $\mathcal{S}^{(\ell)}$</u>. Fix $\ell \in \mathbb{Z}_{\geq 0}$.

Consider Weyl characters as functions only on

$$\mathring{\mathfrak{h}}_\ell = \left\{ e^{-\frac{2\pi i}{\ell+h^\vee}\nu^{-1}(\gamma+\rho)} \;\middle|\; \gamma \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}} \right\}$$

(i.e. $x_1, \dots, x_n$ take values only in $\left\{ e^{\frac{2\pi i}{\ell+h^\vee} j} \;\middle|\; 0 \leq j \leq \ell+h^\vee-1 \right\}$).

The algebra of Weyl characters becomes

$$\mathcal{S}^{(\ell)} = \mathrm{span}\{ s_\lambda \mid \lambda \in P_\ell^+ \}$$

with structure constants the <u>fusion coefficients</u> $N_{\mu\nu}^\lambda$,

$$s_\mu s_\nu = \sum_{\lambda \in P_\ell^+} N_{\mu\nu}^\lambda s_\lambda .$$

Define a $\mathbb{C}$-linear map (a non.deg trace functional)

$$\vec{t} : \mathcal{S}^{(\ell)} \to \mathbb{C} \quad \text{by} \quad \vec{t}(s_\lambda) = \delta_{\lambda 0}$$

and a symmetric bilinear form

$$\langle , \rangle : \mathcal{S}^{(\ell)} \times \mathcal{S}^{(\ell)} \to \mathbb{C} \quad \text{by} \quad \langle a, b \rangle = \vec{t}(ab)$$

Let $\lambda^* = -w_0\lambda$, where w_0 is the "longest element of W_0".

$\{ s_{\lambda^*} \mid \lambda \in P_\ell^+ \}$ is the dual basis to $\{ s_\lambda \mid \lambda \in P_\ell^+ \}$ with respect to $\langle , \rangle$. The irreducible representations of $\mathcal{S}^{(\ell)}$ are

$$\chi^\eta : \mathcal{S}^{(\ell)} \to \mathbb{C}$$

$$s_\lambda \mapsto s_\lambda\left(e^{-\frac{2\pi i}{\ell+h^\vee}\nu^{-1}(\eta+\rho)} \right)$$

Categorification of S

$\mathring{\mathfrak{g}}$ a fin. dim'l reductive Lie algebra over $\mathbb{C}$.

$\langle , \rangle : \mathring{\mathfrak{g}} \times \mathring{\mathfrak{g}} \to \mathbb{C}$ a nondegenerate ad-invariant symmetric bilinear form.

$\mathring{\mathfrak{h}} \subseteq \mathring{\mathfrak{g}}$ a Cartan subalgebra, $\mathring{\mathfrak{h}}^* = \mathrm{Hom}_{\mathbb{C}}(\mathring{\mathfrak{h}}, \mathbb{C})$.

The restriction $\langle , \rangle : \mathring{\mathfrak{h}} \times \mathring{\mathfrak{h}} \to \mathbb{C}$ is nondegenerate

and provides
$$\begin{aligned} \nu : \mathring{\mathfrak{h}} &\xrightarrow{\sim} \mathring{\mathfrak{h}}^* \\ \gamma &\longmapsto \langle \gamma, \cdot \rangle . \end{aligned}$$

$$\left\{ \begin{matrix} \text{irreducible} \\ \text{fin. dim'l} \\ \mathring{\mathfrak{g}}\text{-modules} \end{matrix} \right\} = \left\{ \begin{matrix} \text{irreducible} \\ \text{integrable} \\ \mathring{\mathfrak{g}}\text{-modules} \end{matrix} \right\} \longleftrightarrow \mathring{\mathfrak{h}}^*_{\mathbb{Z}_{\geq 0}} = \mathbb{Z}_{\geq 0}\omega_1 + \cdots + \mathbb{Z}_{\geq 0}\omega_n$$

$$\mathring{L}(\lambda) \longleftarrow\!\shortmid \lambda$$

As $\mathring{\mathfrak{h}}$-modules
$$\mathring{L}(\lambda) = \bigoplus_{\gamma \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}} \mathring{L}(\lambda)_\gamma, \quad \text{where } \mathring{\mathfrak{h}}^*_{\mathbb{Z}} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$$

and $\mathring{L}(\lambda)_\gamma = \{ m \in \mathring{L}(\lambda) \mid hm = \gamma(h) m \text{ for } h \in \mathring{\mathfrak{h}} \}$.

Then
$$s_\lambda = \sum_{\gamma \in \mathring{\mathfrak{h}}^*_{\mathbb{Z}}} \dim(\mathring{L}(\lambda)_\gamma)\, e^\gamma .$$

The <u>dual</u> of $\mathring{L}(\lambda)$ is $\mathring{L}(\lambda)^* = \mathrm{Hom}_{\mathbb{C}}(\mathring{L}(\lambda), \mathbb{C})$ and

$$\mathring{L}(\lambda)^* \simeq \mathring{L}(-w_0\lambda), \quad \text{as } \mathring{\mathfrak{g}}\text{-modules}.$$

Categorification of $S^{(\ell)}$

The *affine Lie algebra* is

$$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d \quad \text{with}$$

$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d \quad \text{and} \quad \mathfrak{h}^* = \mathbb{C}\delta \oplus \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$$

Define $P_\ell^+ \subseteq \mathring{\mathfrak{h}}^*_{\mathbb{Z}_{\geq 0}}$ so that

$$\left\{\begin{matrix}\text{irreducible integrable} \\ \mathfrak{g}\text{-modules with} \\ K \text{ acting by } \ell \cdot \mathrm{Id}\end{matrix}\right\} \longleftrightarrow \mathbb{C}\delta + P_\ell^+ + \ell\Lambda_0$$

$$L(a\delta + \lambda + \ell\Lambda_0) \longleftarrow\!\!\shortmid \ a\delta + \lambda + \ell\Lambda_0$$

Let $\quad \mathcal{U} = \mathbb{P}^1 \setminus \{x_0, x_1, x_2\}$ and $\mathcal{O}[\mathcal{U}] = \left\{\begin{matrix}\text{regular functions} \\ f : \mathcal{U} \to \mathbb{C}\end{matrix}\right\}$

Define an action of

$\mathring{\mathfrak{g}} \otimes \mathcal{O}[\mathcal{U}]$ on $L(\mu + \ell\Lambda_0) \otimes L(\nu + \ell\Lambda_0) \otimes L(\lambda^+ + \ell\Lambda_0)$ by

$$(x \otimes f)(v_1 \otimes v_2 \otimes v_3) = (x \otimes (f)_{x_0}) v_1 \otimes v_2 \otimes v_3 + v_1 \otimes (x \otimes (f)_{x_1}) v_2 \otimes v_3 + v_1 \otimes v_2 \otimes (x \otimes (f)_{x_3}) v_3.$$

where $(f)_{x_i}$ is the Laurent series expansion of f at x_i.

The *space of conformal blocks* is

$$\mathcal{V}^+_{\mu, \nu, \lambda^+}(\mathbb{P}^1) = \mathrm{Hom}_{\mathring{\mathfrak{g}} \otimes \mathcal{O}[\mathcal{U}]}\left(L(\mu + \ell\Lambda_0) \otimes L(\nu + \ell\Lambda_0) \otimes L(\lambda^+ + \ell\Lambda_0), \mathbb{C}\right)$$

and the *fusion coefficients* are

$$N_{\mu\nu}^{\lambda} = \dim\left(\mathcal{V}^+_{\mu, \nu, \lambda^+}(\mathbb{P}^1)\right).$$

Conformal blocks of higher genus

Tsuchiya-Ueno-Yamada used $\mathfrak{g}$-modules to construct conformal field theories

$$\mathcal{V}^{+} : \left\{ \begin{matrix} \text{moduli space of} \\ \text{curves with marked} \\ \text{points labeled by} \\ \text{elements of } P_\ell^{+} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{vector} \\ \text{spaces} \end{matrix} \right\}$$

$$\begin{pmatrix} \mathfrak{X} = (C; Q_1, \dots Q_N) \\ \lambda_1, \dots, \lambda_N \in P_\ell^{+} \end{pmatrix} \longmapsto \mathcal{V}^{+}_{\lambda_1, \dots \lambda_N}(\mathfrak{X})$$

The $\mathcal{V}^{+}_{\lambda_1, \dots \lambda_N}(\mathfrak{X})$ are the spaces of conformal blocks

The fusion coefficients are

$$N_{\mu\nu}^{\lambda} = \dim\left(\mathcal{V}^{+}_{\mu, \nu, \lambda^{+}}(\mathbb{P}^1) \right)$$

The general case:

Fix $\mathfrak{X} = (C; Q_1, \dots, Q_N)$ and $\lambda_1, \dots, \lambda_N \in P_\ell^{+}$. Let

$$g = \text{genus}(C) \quad \text{and} \quad Cas = \sum_{\mu \in P_\ell^{+}} s_\mu s_{\mu^{+}} \quad \text{in } \mathcal{S}^{(\ell)}.$$

Then

$$\dim(\mathcal{V}^{+}_{\lambda_1, \dots, \lambda_N}(\mathfrak{X})) = N_g(\lambda_1 + \cdots + \lambda_N)$$

$$= \vec{t}(s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_N} Cas^{g}) = Tr(s_{\lambda_1} \cdots s_{\lambda_N} Cas^{g-1})$$

$$= \sum_{\eta \in \mathfrak{h}^{0}_{\ell}} \chi^{\eta}(s_{\lambda_1}) \chi^{\eta}(s_{\lambda_2}) \cdots \chi^{\eta}(s_{\lambda_N}) \chi^{\eta}(Cas)^{g-1}$$

where $Tr : \mathcal{S}^{(\ell)} \to \mathbb{C}$ is the trace of the regular representation of $\mathcal{S}^{(\ell)}$.